PB83-207373

Mathematical Models Associated with Point Line Source Discharges in Rivers

(U.S.) Environmental Monitoring and Support Lab.
Cincinnati, OH

May 83

U.S. Department of Commerce
National Technical Information Service
NTIS

TYPING GUIDE SHEET

PB83-207373

EPA-600/4-83-021
May 1983

MATHEMATICAL MODELS ASSOCIATED WITH POINT AND LINE SOURCE DISCHARGES IN RIVERS

by

Philip C. L. Lin, Ph.D.
Physical and Chemical Methods Branch
Environmental Monitoring and Support Laboratory
Cincinnati, Ohio 45268

ENVIRONMENTAL MONITORING AND SUPPORT LABORATORY
OFFICE OF RESEARCH AND DEVELOPMENT
U.S. ENVIRONMENTAL PROTECTION AGENCY
CINCINNATI, OHIO 45268

EPA-287 (Cin.)
(4-76)

REPRODUCED BY
NATIONAL TECHNICAL
INFORMATION SERVICE
U.S. DEPARTMENT OF COMMERCE
SPRINGFIELD, VA. 22161

TECHNICAL REPORT DATA (Please read Instructions on the reverse before completing)		
1. REPORT NO. EPA-600/4-83-021	2.	3. RECIPIENT'S ACCESSION NO. PB83 207373
4. TITLE AND SUBTITLE Mathematical Models Associated With Point and Line Source Discharges in Rivers		5. REPORT DATE May 1983 6. PERFORMING ORGANIZATION CODE
7. AUTHOR(S) Philip C. L. Lin		8. PERFORMING ORGANIZATION REPORT NO.
9. PERFORMING ORGANIZATION NAME AND ADDRESS U. S. Environmental Protection Agency 26 W. St. Clair Street Cincinnati, Ohio 45268		10. PROGRAM ELEMENT NO. A38B1D 11. CONTRACT/GRANT NO.
12. SPONSORING AGENCY NAME AND ADDRESS Environmental Monitoring and Support Laboratory--Cin,OH Office of Research and Development U.S. Environmental Protection Agency Cincinnati, OH 45268		13. TYPE OF REPORT AND PERIOD COVERED Research Report 1980-81 14. SPONSORING AGENCY CODE EPA 600/6
15. SUPPLEMENTARY NOTES		

16. ABSTRACT

A literature search was conducted in the area of mass dispersion from instantaneous and continuous sources in bounded and unbounded flows. This revealed that most of the studies are concentrated on flows in channels with simple geometries such as constant depth and width, and with constant fluid velocity and dispersion coefficients. Very often the depth-averaged value of concentration rather than point value is assumed to approximate the complicated nature of mixing, especially in bounded flows in natural rivers. Analytical solutions often require tedious and repetitious calculations. This study presents a mathematical model consisting of only two nondimensionalized variables and capable of generating universal concentration profiles of the pollutants in a rectangular channel flow for specified initial and boundary conditions. Determination of diffusion coefficients, mixing distance, sampling location and number of sampling points in a cross-section is possible.

In other aspects, determination of the minimum sampling frequency required to collect a representative composite sample based on the previous effort is reexamined. It is concluded that a minimum sampling frequency of eight (8) for most of the flow and concentration patterns in the complete mixing zone is sufficient to provide enough accuracy for a representative sample collected by three out of four known compositing techniques.

17. KEY WORDS AND DOCUMENT ANALYSIS		
a. DESCRIPTORS	b. IDENTIFIERS/OPEN ENDED TERMS	c. COSATI Field/Group
18. DISTRIBUTION STATEMENT Release to Public	19. SECURITY CLASS (This Report) Unclassified 20. SECURITY CLASS (This page) Unclassified	21. NO. OF PAGES 56 22. PRICE

NOTICE

This document has been reviewed in accordance with U.S. Environmental Protection Agency policy and approved for publication. Mention of trade names or commercial products does not constitute endorsement or recommendation for use.

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

FOREWORD

Environmental measurements are required to determine the quality of ambient waters and the character of waste effluents. The Environmental Monitoring and Support Laboratory - Cincinnati conducts research to:

- Develop and evaluate techniques to measure the presence and concentration of physical, chemical, and radiological pollutants in water, wastewater, bottom sediments, and solid waste.
- Investigate methods for the concentration, recovery, and identification of viruses, bacteria, and other microbiological organisms in water. Conduct studies to determine the responses of aquatic organisms to water quality.
- Conduct an Agency-wide quality assurance program to assure standardization and quality control of systems for monitoring water and wastewater.

The function of the Sampling and Field Measurement Section of the Physical and Chemical Methods Branch in the Environmental Monitoring and Support Laboratory is to provide field measurement and sampling techniques relating to water quality research. This report on the investigation of the characteristics of mass dispersion in open channel flows provides mathematical models to be used in monitoring municipal and industrial wastewaters.

Robert L. Booth

Robert L. Booth, Acting Director
Environmental Monitoring and Support Laboratory - Cincinnati

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

ABSTRACT

A literature search was conducted in the area of mass dispersion from instantaneous and continuous sources in bounded and unbounded flows. This revealed that most of the studies are concentrated on flows in channels with simple geometries such as constant depth and width, and with constant fluid velocity and dispersion coefficients. Very often the depth-averaged value of concentration, rather than point value is assumed to approximate the complicated nature of mixing, especially in bounded flows in natural rivers. Analytical solutions often require tedious and repetitious calculations. This study presents a mathematical model consisting of only two nondimensionalized variables and capable of generating universal concentration profiles of the pollutants in a rectangular channel flow for specified initial and boundary conditions. Determination of diffusion coefficients, mixing distance, sampling location and number of sampling points in a cross-section is possible.

In other aspects, determination of the minimum sampling frequency required to collect a representative composite sample based on the previous effort is reexamined. It is concluded that a minimum sampling frequency of eight (8) for most of the flow and concentration patterns in the complete mixing zone is sufficient to provide enough accuracy for a representative sample collected by three out of four known compositing techniques.

This report covers a period from 1965 to 1982, and work was completed as of October 1982.

BEGIN LAST LINE OF TEXT HERE
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
9-1/8"
3/8"

PAGE NUMBER
EPA-287 (Cin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD BEGIN SECTIONS HERE

CONTENTS

Foreword.....iii

Abstract.....iv

Figures.....vi

Tables.....vii

1. Introduction..... 1

2. Conclusions and Recommendations..... 3

3. Mass Dispersion in Open Channel Flows..... 7

4. Development of a Mathematical Model for Mass Dispersion in a Rectangular Channel Flow..... 20

5. Sampling Frequency..... 36

References..... 47

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

PAGE NUMBER
EPA-287 (Cin.)
(4-76)

TYPING GUIDE SHEET
BEGIN FIRST LINE OF TEXT HERE
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

FIGURES

Number		Page
1	Trapezoidal and rectangular channels (see reference 3).....	16
2	Calculated concentration distributions for trapezoidal and rectangular channels with constant dispersion coefficients.....	17
3	X* -η plane in a rectangular channel.....	24
4	Initial and boundary conditions in a rectangular channel flow.....	24
5	Initial and boundary conditions for the half of a rectangular channel flow.....	26
6	Initial and boundary conditions in nondimensionalized form in a rectangular channel flow.....	26
7	Concentration distributions at X* = 0.01, 0.02, 0.03, 0.04 and 0.05 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.....	30
8	Concentration distributions at X* = 0.06, 0.07, 0.08, 0.09 and 0.1 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.....	31
9	Concentration distributions at X* = 0.12, 0.14, 0.16, 0.18 and 0.20 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.....	32
10	Concentration distributions at X* = 0.1, 0.2, 0.3, 0.4 and 0.5 from a line source located at X* = 0 and η = 0 with Re S_{xz} >> 100.....	33
11	Concentration distributions at X* = 0.06, 0.07, 0.08, 0.09 and 0.1 from a line source located at X* = 0 and η = 0 with Re S_{xz} = 100.....	34
12	Concentration distributions at X* = 0.1, 0.2, 0.3, 0.4 and 0.5 from a line source located at X* = 0 and η = 0 with Re S_{xz} = 100.....	35
13	Concentrations of total trihalomethanes in drinking water (see reference 11).....	46

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

PAGE NUMBER
EPA-287 (Cin.) (4-75)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD. BEGIN SECTIONS HERE

TABLES

Number		Page
1	Constants used in Eq. (4.22)	28
2	Results of bulk concentration from analytical and numerical solutions	28
3	Ratio of composite sample concentration to actual concentration (see reference 10)	39
4	Ratio of composite sample concentration to actual concentration in example 1 (method 1)	40
5	Ratio of composite sample concentration to actual concentration in example 1 (method 2)	41
6	Ratio of composite sample concentration to actual concentration in example 1 (method 3)	42
7	Ratio of composite sample concentration to actual concentration in example 1 (method 4)	43
8	Recorded concentrations of total trihalomethanes in drinking water	44
9	Ratio of composite sample concentration to actual concentration in example 2	45

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

PAGE NUMBER
EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD. BEGIN SECTIONS HERE

SECTION 1

INTRODUCTION

When pollutants are discharged into a river, it is often desirable to be able to predict their dispersion downstream from the release point. This results from the importance of obtaining representative samples for any monitoring or enforcement program. Hence, the more accurately one can predict the behavior of the pollutant dispersion, the more representative a sample one can expect to collect.

For the prediction of the pollutant dispersion, one must know the hydraulics of the stream, the physical, chemical, or biological change, and the rate of the pollutant mixing. Because of the complexities of variations of depth and width in most natural rivers, fluid velocity distribution, and the rate of mixing may vary from one location to another. It is, therefore, difficult to fully describe the mixing phenomenon in natural rivers. The approach generally used to predict pollutant mixing in a stream is based on assumptions that an average velocity, an average depth and even a constant width for the river are adopted to characterize the flow in an irregularly shaped water channel. Many unknown aspects of the mixing in the flow are then implicitly lumped into coefficients of dispersion which are, in turn, used for the prediction of pollutant dispersion. A review of the literature reveals that most of the analytical solutions of pollutant mixing are either for instantaneous point and line sources in an unbounded fluid or for continuous line sources in both unbounded and bounded fluid. Among the two categories of sources, the instantaneous sources are only of theoretical interest in a practical sense. The continuous sources, however, provide a deeper insight into the behavior of the mass being transported, as actual pollutant spills are never instantaneous, but are spread over a finite time interval. Although analytical solutions on the prediction of pollutant mixing are closed forms, they often require tedious and time-consuming calculations. Furthermore, repetitious calculations are needed for any change of values in variables such as velocity, depth, or dispersion coefficients. Most of the solutions also have the aforementioned restrictions such as constant channel width and a constant velocity. It is the intention in this study to develop a mathematical model which combines variables such as distance, dispersion coefficients, depth, and kinematic viscosity into groups of nondimensionalized variables and, therefore, to produce universal concentration profiles for the river pollutants under specified conditions in the area of mass dispersion. This model also has its limitations and advantages. It is restricted to a rectangular channel flow, however, is allowed to have a variable velocity profile across a section of the channel. Also included is a further examination of the

BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
BEGIN LAST LINE OF TEXT
3/8"

EPA-207 (Cin.)
(4-76)

TYPING GUIDE SHEET
BEGIN FIRST LINE OF TEXT HERE
CENTER OF PAGE
TOP OF IMAGE AREA
minimum sample frequency for collecting representative samples under completely mixed conditions for variable flow and concentration patterns.
DROPPED HEAD, BEGIN SECTIONS HERE
6-1/2"
9-1/8"
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
BEGIN LAST LINE OF TEXT
3/8"
2
PAGE NUMBER
EPA-287 (Cin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTION HERE

SECTION 2

CONCLUSIONS AND RECOMMENDATIONS

Most predictions of concentration distributions from point or line sources in bounded or unbounded flows are expressed as closed-form solutions under conditions of uniform or average flow velocity, constant channel width or dispersion coefficients. Only those solutions from continuous sources in bounded flows seem applicable to simulate problems in rivers. Conclusions drawn from the literature search in this area are summarized as follows:

- For flow in a rectangular channel with constant flow velocity and constant dispersion coefficient in the transverse direction:

  Eq. (3.26) or (3.27) in Section 3 may be used to evaluate the concentration distributions at any location of the river. Eq. (3.30), also in Section 3, may be used to determine the mixing distance where the material being transported in the channel is completely mixed. Eqs. (3.26), (3.27), and (3.30) are presented as follows:

$$C(x,z) = \frac{Q_m}{h(\pi e_z u x)^{\frac{1}{2}}} \sum_{n=-\infty}^{n=\infty} \exp\left\{-\frac{u(z-2nH)^2}{4e_z x}\right\} \qquad (3.26)$$

$$C(x,z) = \frac{Q_m}{hHu} + \frac{2Q_m}{hHu} \sum_{n=1}^{n=\infty} \frac{\cos\frac{n\pi z}{H}}{(1+\alpha_n)^{\frac{1}{2}}} \exp\left\{\frac{ux}{2e_z}\left(1-(1+\alpha_n)^{\frac{1}{2}}\right)\right\} \qquad (3.27)$$

$$X_m = \frac{0.445H^2 u}{e_z} \qquad (3.30)$$

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Clin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD BEGIN SECTIONS HERE

where

C = concentration of material being transported in the channel
x = coordinate in the flow direction in the channel
z = coordinate in the transverse direction in the channel
Q_m = a constant release of material emanating at the origin of the coordinates.
h = depth of the channel
e_z = dispersion coefficient in the transverse direction, z, in the channel
u = flow velocity
H = channel width
$\alpha_n = \left(\frac{2e_z n\pi}{uH} \right)^2$
X_m = mixing distance

- For flow in a river with variable depth:

Eq. (3.38) in Section 3 can be used to calculate the concentration distributions if the distributions of flow velocity, river depth, and dispersion coefficient across the river in any section are known. Eq. (3.38) is presented as follows:

$$C'(\alpha,q') = \frac{1}{2(q'_{s2}-q'_{s1})} \left[\sum_{n=o}^{n=\infty} \left\{ \operatorname{erf} \frac{\alpha(q'_{s2}+2n-q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}+2n-q')}{\sqrt{2}} + \operatorname{erf} \frac{\alpha(q'_{s2}+2n+q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}+2n+q')}{\sqrt{2}} \right\} + \sum_{n=o}^{n=\infty} \left\{ \operatorname{erf} \frac{\alpha(q'_{s2}-2n-q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}-2n-q')}{\sqrt{2}} + \operatorname{erf} \frac{\alpha(q'_{s2}-2n+q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}-2n+q')}{\sqrt{2}} \right\} \right] \quad (3.38)$$

where

$$\operatorname{erf} q = \frac{2}{\sqrt{\pi}} \int_0^q \exp(-p^2)\, dp$$

$$\alpha = \frac{Q^2}{2x\, e_z u h^2}$$

BOTTOM OF IMAGE AREA. OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
BEGIN LAST LINE OF TEXT
3/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

$$q' = \frac{q}{Q}$$

$$C' = \frac{C}{\frac{1}{Q}\int_0^Q C\, dq}$$

$$q = \int_0^z uh\, dz$$

$$Q = \int_0^H uh\, dq$$

q'_{s1}, q'_{s2} = the two end points of the line source expressed as fractions of the total discharge Q

This study presents simple universal concentration profiles for a rectangular channel flow. The profiles are valid for steady-state conditions from continuous sources in the stream. The applications for this model can be summarized as follows:

- Determination of Schmidt number, S_z, and dispersion coefficient, e_z:

  The concentration distributions across a water channel at any downstream location X are measured and also are obtained from the model. Comparison of the results will indicate that an appropriate value of nondimensionalized distance X^* should be selected. Therefore, Schmidt number, S_z, and dispersion coefficient, e_z can be determined as follows:

$$S_z = \frac{X^* \, ReH}{X} \qquad (2.1)$$

where

$$Re = \frac{1.5\ UH}{\nu}, \quad \text{Reynolds number}$$

ν = kinematic viscosity of water
U = average velocity in the channel
H = channel width

$$e_z = \frac{\nu}{S_z} \qquad (2.2)$$

- Determination of sampling location X:

  Once the Schmidt number is known, one can choose a known concentration distribution of X^* and locate the sampling location X as follows:

$$X = X^* \, Re \, S_z \, H \qquad (2.3)$$

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

- Determination of number of sampling points at location X:

  Since the concentration distributions are known at a certain value of X*, the value of X can be calculated. The number of sampling points are then determined from the known concentration distributions by Simpson's rule or any other approximate technique.

- Determination of the mixing distance X_m:

  The mixing distance for a continuous discharge in a rectangular channel is

$$X_m = K\, Re S_z H \qquad (2.4)$$

  where

  K = 0.14 for a continuous discharge at one of the banks
  K = 0.30 for a continuous discharge in the center of the channel.

Although this effort is restricted to a rectangular channel flow, it also provides an approximate solution for channel flows with constant width and moderate variations of depth. This study is a continuous effort in this area and will be helpful to field personnel in selecting the sampling location and number of sampling points.

In the study of sampling frequency, it is found that a minimum sampling frequency of m = 8 is sufficient to provide enough accuracy for samples being collected for most of the flow and concentration patterns studied. The capability to analytically solve the mixing phenomenon in rivers is very limited, as can be seen from the literature being reviewed. The present mathematical model which includes velocity variations also has its limitations, such as the restrictions of constant channel width and depth. There is therefore a need to develop a model which takes into consideration variations of depth and width and velocity in a river. The possible solution may only be numerically obtained through the application of a computer.

9-1/8"
BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

PAGE NUMBER
EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE
6-1/2"

SECTION 3

MASS DISPERSION IN OPEN CHANNEL FLOWS

The prediction of mass dispersion in a flow may be expressed in a mathematical form which represents the equation of conservation of mass

$$\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = D \frac{\partial^2 C}{\partial x_i \partial x_i} + r_a \qquad (3.1)$$

where

C = concentration of mass being dispersed
u_i = component of fluid velocity in the i direction
x_i = coordinate in the i direction
D = molecular diffusivity of mass being dispersed in the fluid
r_a = rate of production or decay of mass being dispersed in the fluid
t = time

The second and third terms in Eq. (3.1) represent the convective and diffusive transport, respectively, which cross the boundaries of the control volume being investigated. The last term is the rate of production or decay of the mass within the control volume. Finally, the first term is the overall rate of change of mass concentration. The solution to Eq. (3.1) displays the concentration distributions of material being transported in a stream in space and time. In laminar flow, the various quantities, such as concentration and fluid velocity, may be expressed explicitly in time and space in the flow field. In turbulent flow, the fluid motion is an irregular conditon of flow in which various quantities show random variations with time and space coordinates and never reach a truly steady-state condition. Therefore, no exact solution to Eq. (3.1) exists in turbulent flow. Unfortunately, most flows in natural rivers are turbulent and the prediction of the flow velocity and concentration is very difficult. Statistically, distinct average values in a turbulent flow can be discerned. It is generally agreed that the description of a turbulent flow in a mathematical form is to introduce time-averaged values, $\overline{u}_i$ or $\overline{C}$, and the fluctuating components, u_i' or C' into Eq. (3.1). The velocity and concentration may be, therefore, written as:

$$u_i = \overline{u}_i + u_i' \qquad (3.2a)$$

$$C = \overline{C} + C' \qquad (3.2b)$$

BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
BEGIN LAST LINE OF TEXT
3/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD BEGIN SECTIONS HERE

Substitute Eq. (3.2) into Eq. (3.1) and average it over a time period. This time period must be large compared with the time scale of the turbulence and small compared with the period of any slow variations in flow fields not belonging to the turbulence. After deleting the "r_a" term for conservative material, Eq. (3.1) becomes:

$$\frac{\partial \bar{C}}{\partial t} + \bar{u}_i \frac{\partial \bar{C}}{\partial x_i} = \frac{\partial}{\partial x_i} \left(D \frac{\partial \bar{C}}{\partial x_i} - \overline{u_i' C'} \right) \qquad (3.3)$$

The term, $\overline{u_i'C'}$, represents the average value of the product of two fluctuations, u_i' and C', and may be represented as:

$$\overline{u_i' C'} = - \varepsilon_{Di} \frac{\partial \bar{C}}{\partial x_i} \qquad (3.4)$$

Substituting Eq. (3.4) into (3.3), Eq. (3.3) with the average "bar" dropped for simplicity becomes:

$$\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\varepsilon_i \frac{\partial C}{\partial x_i} \right) \qquad (3.5)$$

where

ε_{Di} = turbulent diffusivity of mass being dispersed in the fluid in the i direction

$\varepsilon_i = D + \varepsilon_{Di}$

Eq. (3.5) is applicable to turbulent as well as laminar flow in which the turbulent diffusivities, ε_{Di}, diminish. The solution to Eq. (3.5) represents a complete concentration distribution in time and space coordinates in the flow field. The solution provides sufficient information for the determination of the number of sampling points in a location required to collect representative samples. Unfortunately, most natural rivers have irregular geometrics so that solutions for Eq. (3.5) are difficult. Nonetheless, solutions for appropriate initial and boundary conditions for certain types of flow are available. The knowledge of these solutions as background information is important for pursuing further study of mass dispersion problems in natural rivers.

A literature search has been conducted and some selected solutions to Eq. (3.5) are presented as follows:

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"
6-1/2"
3-1/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

I. INSTANTANEOUS POINT SOURCES IN AN UNBOUNDED FLUID

Study of instantaneous point sources of an unbounded fluid is only of academic interest. Nonetheless, some fundamental solutions may still provide an insight to the material dispersion in the fluid. Solutions are divided into two groups. One is for fluid with zero velocity, the other is for fluid with non-zero velocity.

1. FLUID WITH ZERO VELOCITY

Eq. (3.5) in Cartesian coordinates can be written as:

$$\frac{\partial C}{\partial t} = D \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} \right) \tag{3.6}$$

Three solutions under one, multiple and continuous point sources are available and are presented below

a. Solutions for one instantaneous point source at the origin of the coordinates:

For a three-dimensional fluid:

$$C(x,y,z,t) = \frac{M}{8(\pi Dt)^{3/2}} \exp \left(- \frac{x^2+y^2+z^2}{4Dt} \right) \tag{3.7}$$

For a two-dimensional fluid:

$$C(x,y,t) = \frac{M}{4\pi Dt} \exp \left(- \frac{x^2+y^2}{4Dt} \right) \tag{3.8}$$

For a one-dimensional fluid:

$$C(x,t) = \frac{M}{2(\pi Dt)^{1/2}} \exp \left(- \frac{x^2}{4Dt} \right) \tag{3.9}$$

where M is the amount of material emanating instantaneously at $t = 0$. These solutions clearly show that the concentration always peaks at the origin of the coordinates. In time, the concentration distributions spread over the space and become more uniformly distributed.

b. Solutions for N instantaneous point sources located at (x_i, y_i, z_i) and discharges commencing at $t = t_i$, $i = 1, 2, 3, \ldots, N$:

For a three-dimensional fluid:

$$C(x,y,z,t) = \frac{1}{8(\pi D)^{3/2}} \sum_{i=1}^{i=N} \frac{M_i}{(t-t_i)^{3/2}} \exp \left\{ - \frac{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}{4Dt} \right\} \tag{3.10}$$

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

For a two-dimensional fluid:

$$C(x,y,t) = \frac{1}{4\pi D} \sum_{i=1}^{i=N} \frac{M_i}{(t-t_i)} \exp\left\{-\frac{(x-x_i)^2+(y-y_i)^2}{4Dt}\right\} \quad (3.11)$$

For a one-dimensional fluid:

$$C(x,t) = \frac{1}{2(\pi D)^{1/2}} \sum_{i=1}^{i=N} \frac{M_i}{(t-t_i)^{1/2}} \exp\left\{-\frac{(x-x_i)^2}{4Dt}\right\} \quad (3.12)$$

where M_i is the amount of material emanating instantaneously at t=0 and at locations (x_i, y_i, z_i), i=1, 2,3,....., N. The concentration distributions in Eqs. (3.10), (3.11), and (3.12) are similar to those in (a). These solutions are superimposed because the partial differential equation is linear.

c. Solutions for a continuous set of instantaneous point source discharge commencing at t=t' and located at (x',y',z'):

For a three-dimensional fluid:

$$C(x,y,z,t) = \frac{1}{8(\pi D)^{3/2}} \int_{t'=o}^{t} \frac{dt'}{(t-t')^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M\, dx'\, dy'\, dz'$$

$$\exp\left\{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4Dt}\right\} \quad (3.13)$$

For a two-dimensional fluid:

$$C(x,y,t) = \frac{1}{4\pi D} \int_{t'=o}^{t} \frac{dt'}{(t-t')} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} M\, dx'\, dy'$$

$$\exp\left\{-\frac{(x-x')^2+(y-y')^2}{4Dt}\right\} \quad (3.14)$$

For a one-dimensional fluid:

$$C(x,t) = \frac{1}{2(\pi D)^{1/2}} \int_{t'=o}^{t} \frac{dt'}{(t-t')^{1/2}} \int_{-\infty}^{\infty} M\, dx' \exp\left\{-\frac{(x-x')^2}{4Dt}\right\} \quad (3.15)$$

where M is the amount of material emanating instantaneously at t=t' and at location (x',y',z').

These solutions are the most general among the three kinds of instantaneous sources. However, to integrate Eqs. (3.13),

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.) (4-76)

(3.14), and (3.15) over time and space is difficult and may not be practical.

2. FLUID WITH NON-ZERO VELOCITY

This type of flow is characteristic of natural rivers. The governing equation for this kind of flow can be written in Cartesian coordinates as:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{\partial}{\partial x}(\varepsilon_x \frac{\partial C}{\partial x}) + \frac{\partial}{\partial y}(\varepsilon_y \frac{\partial C}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon_z \frac{\partial C}{\partial z}) \quad (3.16)$$

where u, v, and w are velocities in the direction of the x, y, and z coordinates, respectively. The solutions with different boundary conditions are presented. The first one has constant u, v=w=o, and constant ε_x, ε_y and ε_z. The second one has variable u, v=w=o, and constant ε_x, ε_y and ε_z.

a. Solution for an instantaneous point source at the origin of the coordinates with constant u=U, v=w=o and constant ε_x, ε_y and ε_z:

The following solution is obtained from Sayre[1]:

$$C(x,y,z,t) = \frac{C_o V_o}{8(\pi t)^{3/2}(\varepsilon_x \varepsilon_y \varepsilon_z)^{1/2}} \exp\left(-\frac{(x-Ut)^2}{4\varepsilon_x t} - \frac{y^2}{4\varepsilon_y t} - \frac{z^2}{4\varepsilon_z t}\right) \quad (3.17)$$

where

C_o = initial concentration at the point of source
V_o = initial volume having concentration C_o
U = constant fluid velocity

b. Solution for an instantaneous point source at the origin of the coordinates with u varying with the cross-section (uniform shear flow), v=w=o and constant ε_x, ε_y and ε_z:

The following solution is obtained from Carter and Okubo[2]:

$$C(x,y,z,t) = \frac{C_o V_o}{8(\pi t)^{3/2}(\varepsilon_x \varepsilon_y \varepsilon_z)^{1/2}(1+\theta^2 t^2)^{1/2}} \exp\left\{-\frac{(x-Ut-\frac{1}{2}(\Gamma_y y+\Gamma_z z)t)^2}{4\varepsilon_x t(1+\theta^2 t^2)} - \frac{y^2}{4\varepsilon_y t} - \frac{z^2}{4\varepsilon_z t}\right\} \quad (3.18)$$

where

$$\theta^2 = \frac{1}{12}\left(\frac{\Gamma_y^2 \varepsilon_y}{\varepsilon_x} + \frac{\Gamma_z^2 \varepsilon_z}{\varepsilon_x}\right)$$

Γ = velocity gradient
U = average velocity in the cross section

The above solutions for specific initial and boundary conditions are valid for an unbounded fluid either in one, two or three dimensions. In fact, no unbounded fluid exists in natural rivers. One may consider the solutions as approximate solutions when suitable conditions are met.

II. INSTANTANEOUS LINE SOURCES IN AN UNBOUNDED FLUID

For many mixing problems in natural rivers, the vertical profile of the concentration is often described by the depth-averaged value of concentration, rather than point value. In order to obtain a two-dimensional equation representing the variation of the depth-averaged concentration in the transverse and longitudinal directions, Eq. (3.5) may be integrated over the water depth. The following equation in Cartesian coordinates is obtained from Holley[3]:

$$\frac{\partial(h\bar{C})}{\partial t} + \frac{\partial}{\partial x}(hu\bar{C}) + \frac{\partial}{\partial z}(hw\bar{C}) = \frac{\partial}{\partial x}(he_x \frac{\partial \bar{C}}{\partial x}) + \frac{\partial}{\partial z}(he_z \frac{\partial \bar{C}}{\partial z}) \qquad (3.19)$$

where

e_x, e_z = dispersion coefficients which include transport due to both turbulent fluctuations and vertical variations of longitudinal and transverse velocity.
h = depth of water

Eq. (3.19) serves as the basis of a variety of analytical and numerical models of mixing in natural streams using a depth-averaged value of concentration. Solutions to Eq. (3.19) for two different mixings from an instantaneous vertical line source with constant depth and dispersion coefficients and w=o are available. Eq. (3.19) is then reduced to:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = e_x \frac{\partial^2 C}{\partial x^2} + e_z \frac{\partial^2 C}{\partial z^2} \qquad (3.20)$$

where the average "bar" is again dropped for simplicity.

a. Solution for an instantaneous line source located at x=0, z=0 and injection of material at t=0 with constant u=U:
For an unbounded two-dimensional flow, Sayre and Chang[4] presented the following solution:

$$C(x,z,t) = \frac{M}{4\pi ht(e_x e_z)^{1/2}} \exp\{-\frac{(x-Ut)^2}{4e_x t} - \frac{z^2}{4e_z t}\} \qquad (3.21)$$

b. Solution for an instantaneous line source located at x=0, z=0 and injection of material at t=0 with u varying linearly with z (uniform shear flow):

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD; BEGIN SECTIONS HERE

Eq. (3.20) becomes

$$\frac{\partial C}{\partial t} + (U_o + \Gamma_z z)\frac{\partial C}{\partial x} = e_x \frac{\partial^2 C}{\partial x^2} + e_z \frac{\partial^2 C}{\partial z^2} \qquad (3.22)$$

where

U_o = velocity at z=0

The solution given by Monin and Yaglom[5] is:

$$C(x,z,t) = \frac{M}{4\pi ht(1+\theta^2 t^2)(e_x e_z)^{1/2}} \exp\left\{-\frac{(x-U_o-\Gamma_z zt)^2}{4e_x t(1+\theta^2 t^2)} - \frac{z^2}{4e_z t}\right\} \qquad (3.23)$$

where

$$\theta^2 = \frac{\Gamma_z e_x}{12 e_z}$$

III. A CONTINUOUS LINE SOURCE IN AN UNBOUNDED FLUID

An instantaneous point or line source is different from a continuous line source in that a constant release of mass eventually causes the concentration distributions to reach steady state in time and space. For two-dimensional mixing, Sayre and Chang[4] integrated Eq. (3.21) for a solution of an instantaneous line source over a sufficiently large time period to obtain a steady state concentration distribution in the x and z coordinates. It may be written as:

$$C(x,z) = \frac{Q_m}{2\pi h(e_x e_z)^{1/2}} \exp\left\{\frac{ux}{2e_x} Ko\left(\frac{u}{2e_x}\left(x^2 + \frac{e_x}{e_z} z^2\right)^{\frac{1}{2}}\right)\right\} \qquad (3.24)$$

The fluid is assumed to be laterally unbounded and Ko is a modified Bessel function of the second kind of zero order. Q_m is a constant release of material emanating at the origin of the coordinates. Other parameters, h, e_x, e_z, and u are constants.

If the dispersion coefficient in the x direction is neglected, the solution is:

$$C(x,z) = \frac{Q_m}{2h\sqrt{\pi e_z ux}} \exp\left(-\frac{uz^2}{4e_z x}\right) \qquad (3.25)$$

The concentrations in both equations become smaller and smaller as the values of x and z increase. However, the total mass across any boundary around the source will be constant and equal to Q_m. For the material released at some points $x-x_o$ $z=z_o$, instead of x=0 z=0, x and z in Eqs. (3.21), (3.24) and (3.25) should be replaced by $(x-x_o)$ and $(z-z_o)$.

BEGIN LAST LINE OF TEXT HERE
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS

EPA-287 (Cin.) (4-76)

IV. CONTINUOUS LINE SOURCES IN BOUNDED FLUID

Flows in natural rivers or water channels are always bounded. Solutions for this type of flow are, therefore, important. The following are three selected solutions, one for a rectangular channel and the other two for channels with variable depths.

1. FLOW IN A RECTANGULAR CHANNEL OF WIDTH, H, WITH CONSTANT u, e_z and $e_x = 0$

By using the principle of images for a continuous release of mass at z=0, Holley[3] obtained a steady state solution:

$$C(x,z) = \frac{Q_m}{h(\pi e_z u x)^{1/2}} \sum_{n=-\infty}^{n=\infty} \exp\left\{-\frac{u(z-2nH)^2}{4e_z x}\right\} \qquad (3.26)$$

where the longitudinal dispersion coefficient is neglected. Carslaw and Jaeger[6] also directly solved the partial differential equation and obtained the following solution:

$$C(x,z) = \frac{Q_m}{hHu} + \frac{2Q_m}{hHu} \sum_{n=1}^{n=\infty} \frac{\cos(\frac{n\pi z}{H})}{(1+\alpha_n)^{1/2}} \exp\left\{\frac{ux}{2e_z}(1-(1+\alpha_n)^{1/2})\right\} \qquad (3.27)$$

where

$$\alpha_n = \left(\frac{2e_z n\pi}{uH}\right)^2$$

When the material being released is completely mixed with the flow, the final uniform concentration, C_∞, is:

$$C_\infty = \frac{Q_m}{hHu} \qquad (3.28)$$

These two solutions, Eqs. (3.26) and (3.27), are numerically equivalent. Both solutions have a continuous line source at one of the banks of the river. By using Eq. (3.26), it is possible to estimate the longitudinal distance required for the material to spread across the channel which according to Holley[3], is defined as the distance X_c. The criterion in defining X_c is the percentage ratio between the concentration difference (C_o (at z=0) – C_H (at z=H) and C_o. Taking the percentage as 2% gives:

$$X_c = \frac{0.0543\ H^2 u}{e_z} \qquad (3.29)$$

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD. BEGIN SECTIONS HERE

Similarly, Eq. (3.27) may be used for estimating the distance required for the mass to completely mix with the flow. The criterion used to define the complete mixing distance, X_m, is to let C_0 and C_H each varying by 2.5% from the uniform concentration C_∞ . Taking the percentage as 2.5% gives:

$$X_m = \frac{0.445\ H^2 u}{e_z} \tag{3.30}$$

Estimates for X_c and X_m are valid only for a rectangular channel with constant u, e_z and $e_x = 0$ and a continuous line source at z=0, x=0. The restriction of e_x=0 is generally valid because the longitudinal dispersion is small as compared with that in the cross section.

2. FLOW IN A VARIABLE DEPTH CHANNEL OF WIDTH H WITH CONSTANT u, e_z and e_x=0

The influence of the variation of depth on the concentration distributions in the stream was investigated by Siemons on two different channels shown in Figure 1 (Holley[3]). The trapezoidal channel has the same cross-sectional area and surface width as the rectangular channel. The depth of the rectangular channel is determined as the ratio of the cross-sectional area and the surface width.

Results of steady state concentration distributions obtained by Siemons are shown in Figure 2 as a result of continuous release of 10^{-4}Kg/sec. at the left bank (z=0) of the channels. As can be seen in Figure 2, there is a rather significant difference between the two results at a small value of x. But as the value of x increases, the two curves of concentration approach each other. It is reasonable to assume that the effect of channels with moderate depth variations further downstream from the source may be neglected. A rough estimate for this distance is about 120 times the depth of the rectangular channel.

Another analytical solution to Eq. (3.19) for nonrectangular geometry of a natural river has been developed by Yotsukura and Cobb[7] using the method of images. For a continuous injection of material from a vertical line source into a stream where the depth, fluid velocity and diffusivity are assumed to be functions of z only, Eq. (3.19) becomes:

$$hu\frac{\partial C}{\partial x} = \frac{\partial}{\partial z}\left(he_z\frac{\partial C}{\partial z}\right) \tag{3.31}$$

where the dispersion coefficient in the longitudinal direction is neglected.

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.)
(4-76)

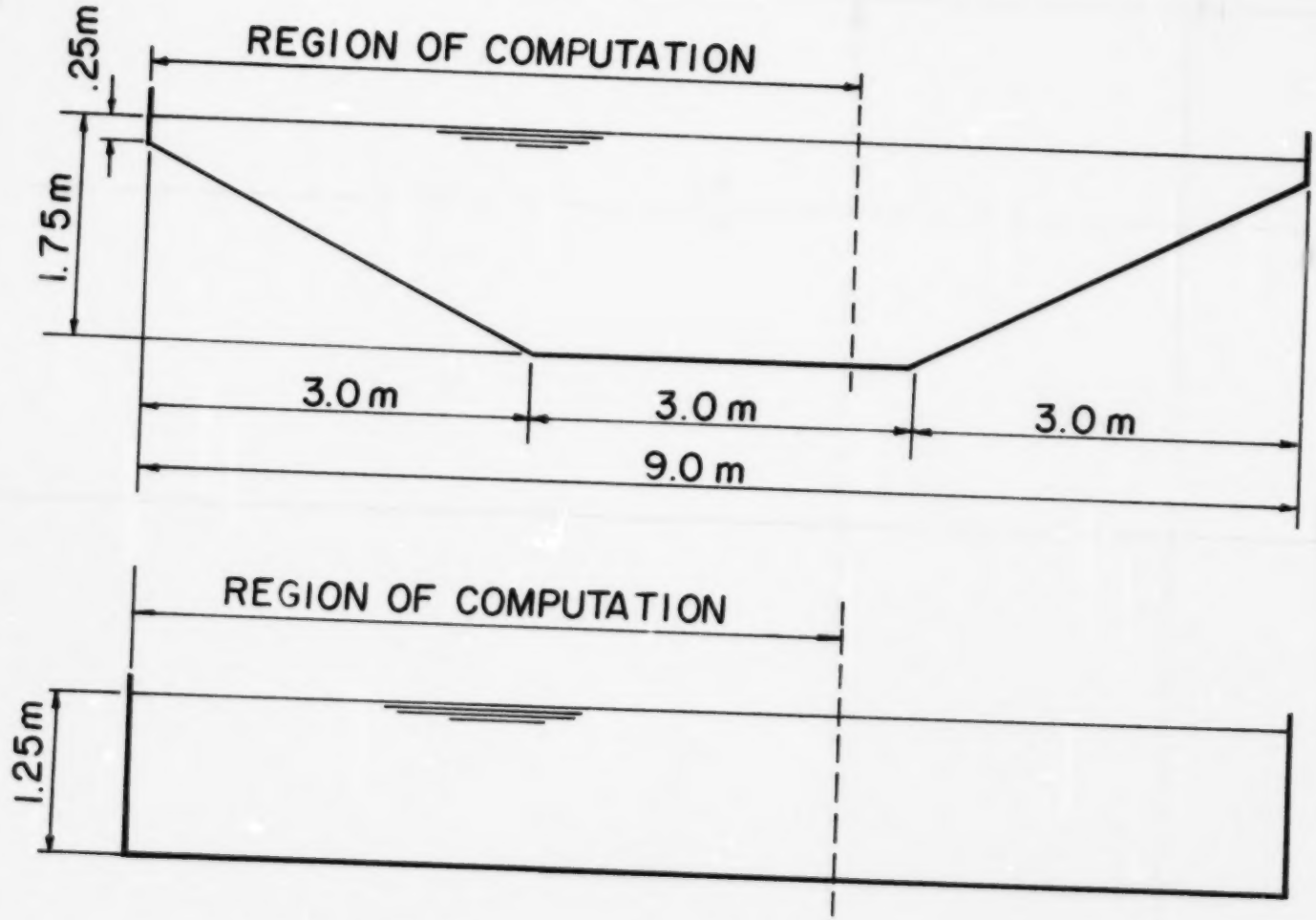

Figure 1. Trapezoidal and rectangular channels (see reference 3).

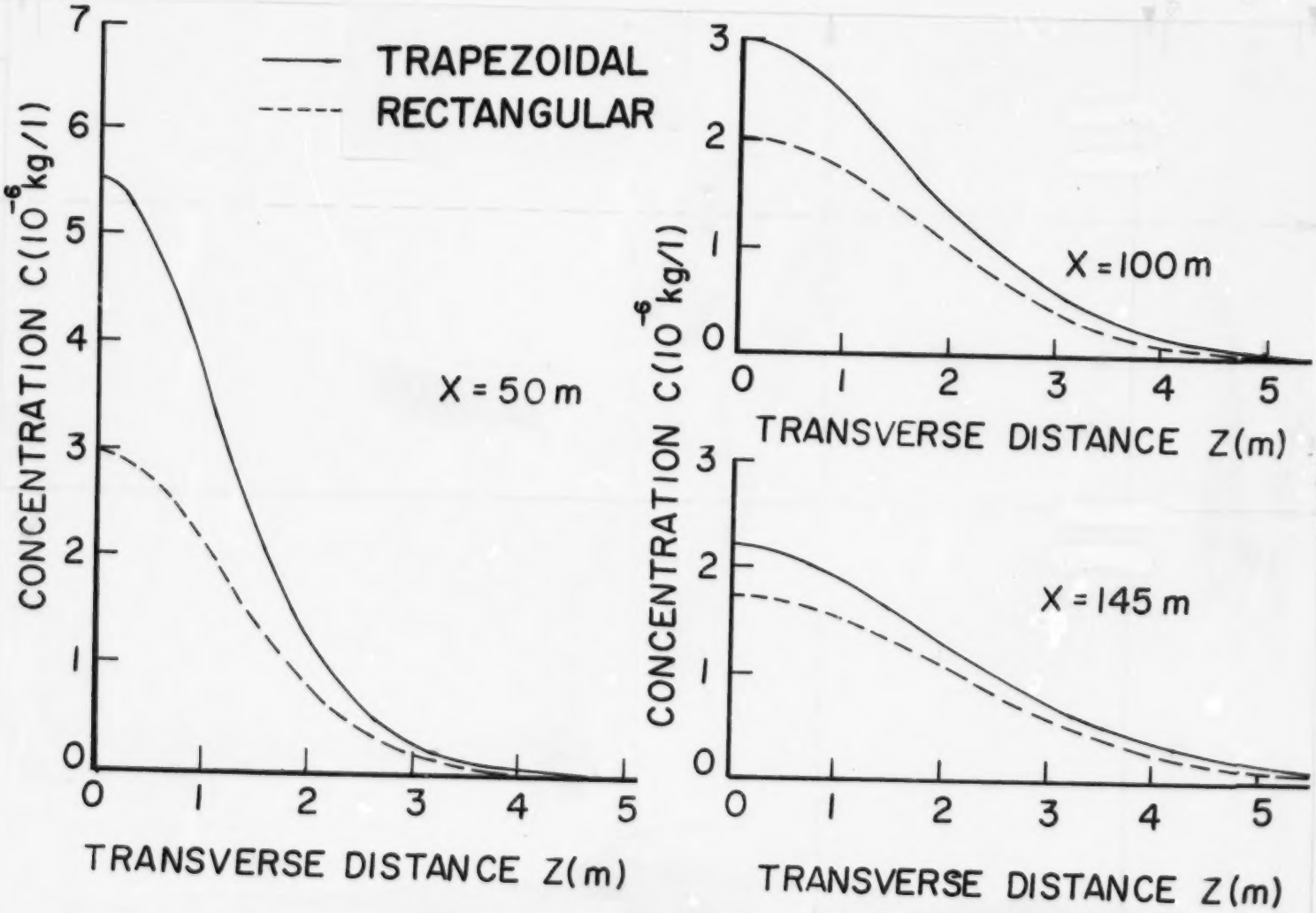

Figure 2. Calculated concentration distributions for trapezoidal and rectangular channels with constant dispersion coefficients.

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD; BEGIN SECTIONS HERE

A closed-form analytical solution is impossible because u and h vary transversely in the stream. By introducing a new variable, q:

$$q = \int_0^z uh\, dz \tag{3.32}$$

they transformed Eq. (3.31) into:

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial q}(e_z uh^2 \frac{\partial C}{\partial q}) \tag{3.33}$$

Again, Eq. (3.33) cannot be solved analytically due to the variable coefficients, $e_z uh^2$. However, by assuming an average constant factor, $\overline{e_z uh^2}$:

$$\overline{e_z uh^2} = \int_0^Q e_z uh^2 dq \tag{3.34}$$

where Q is the total discharge of material in the cross section of the channel.

They replaced Eq. 3.33 by:

$$\frac{\partial C}{\partial x} = \overline{e_z uh^2} \frac{\partial^2 C}{\partial q^2} \tag{3.35}$$

with the following boundary conditions,

$$\frac{\partial C}{\partial q} = 0 \qquad \text{at } q = 0 \text{ and } q = Q \tag{3.36}$$

$$M = \gamma \int_0^Q C\, dq \tag{3.37}$$

where

γ = specfic weight of water

$Q = \int_0^H uh\, dq$

The analytical solution for the above set equations is:

$$C'(\alpha,q') = \frac{1}{2(q'_{s2}-q'_{s1})} \Big[\sum_{n=o}^{n=\infty} \{ \operatorname{erf} \frac{\alpha(q'_{s2}+2n-q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}+2n-q')}{\sqrt{2}} + \operatorname{erf} \frac{\alpha(q'_{s2}+2n+q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}+2n+q')}{\sqrt{2}} \}$$

$$+ \sum_{n=o}^{n=\infty} \{ \operatorname{erf} \frac{\alpha(q'_{s2}-2n-q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}-2n-q')}{\sqrt{2}}$$

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS

EPA-287 (Cin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD. BEGIN SECTIONS HERE

$$+ \operatorname{erf} \frac{\alpha(q'_{s2}-2n+q')}{\sqrt{2}} - \operatorname{erf} \frac{\alpha(q'_{s1}-2n+q')}{\sqrt{2}} \}] \qquad (3.38)$$

where

$$\operatorname{erf} q = \frac{2}{\sqrt{\pi}} \int_0^q \exp(-p^2)\, dp$$

$$\alpha = \frac{Q^2}{2x\,\overline{e_z uh^2}}$$

$$q' = \frac{q}{Q}$$

$$C' = \frac{C}{\frac{1}{Q}\int_0^Q C\, dq}$$

$$q = \int_0^z uh\, dz$$

$$Q = \int_0^H uh\, dq$$

q'_{s1}, q'_{s2} = the two endpoints of the line source expressed as fractions of the total discharge Q.

The calculations of q, Q, and $\overline{e_z uh^2}$ in Eq. (3.38) through integration or numerical estimation with measured velocity profile and depth distributions across the channel may be tedious and time-consuming, however, this equation appears to be applicable for the concentration computation in natural rivers for line or plane source discharges.

In summary, most of analytical solutions obtained so far are only applicable to flows with an average velocity, an average depth and constant dispersion coefficients. Solutions in IV from continuous line sources in bounded fluid, however, may be the most useful among the others.

V. PRESENT STUDY

The present study intends to add another approach in Category IV and to derive a mathematical model which combines variables into groups of nondimensionalized parameters. The solution to this model will provide universal concentration profiles in a rectangular channel flow under specified conditions. This model may also be used to approximate the solution to nonrectangular channels with moderate depth-variations and constant widths, and will not be limited to an average flow velocity often assumed in the analytical solutions.

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

PAGE NUMBER
EPA-287 (Cin.) (4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD; BEGIN SECTIONS HERE

SECTION 4

DEVELOPMENT OF A MATHEMATICAL MODEL FOR MASS DISPERSION IN A RECTANGULAR CHANNEL FLOW

The study of mass dispersion in a fluid from instantaneous sources is only of academic interest, as is pointed out in the previous section; however, it does provide fundamental information about the phenomenon of mass dispersion in the fluid. Molecular diffusivity, dispersion coefficient, and turbulent diffusivity are often determined from the instantaneous point sources. These parameters should be known beforehand so that concentration distribution can be determined.

It would be more practical to study the same problem from continuous line sources in natural rivers. To date, the prediction of concentration distributions from continuous sources are expressed in closed-form solutions under conditions e.g., average fluid velocity, constant channel width, constant dispersion coefficients. They are useful, although most of them are generally based on very fundamental idealized conditions.

The purpose of this study was to develop a mathematical model from which universal concentration profiles for certain types of channels can be obtained in the flow field.

I. THEORETICAL ANALYSIS

It is well known that Eq. (3.19) is the basic equation for a variety of analytical and numerical models of mixing in natural rivers. It describes the conservation of mass being transported in a fluid using the depth-averaged value of concentration instead of the point value. The use of this value has great advantages over that of the point value, mainly because the boundary conditions and geometries of natural rivers may be simplied. Thus, solutions for concentration distributions may be approximated.

In the previous section, it was concluded that the effect on the concentration distributions for channels with moderate depth variations is not significant at distances greater than 120 times the depth of the channel. Since the depth in natural rivers is generally small as compared to their width and the length, the assumption of complete mixing in the vertical direction in a rectangular channel flow may not produce any significant error.

For the development of a mathematical model on the universal concentration profiles, let us assume the channel to be a rectangular one with

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA; OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS
3/8"

EPA-287 (Cin.)
(4-76)

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD: BEGIN SECTIONS HERE

constant dispersion coefficients in the x and z directions. Thus, Eq. (3.19) for a steady state condition can be written as:

$$u \frac{\partial C}{\partial x} = e_x \frac{\partial^2 C}{\partial x^2} + e_z \frac{\partial^2 C}{\partial z^2} \tag{4.1}$$

The coordinates are selected so that x measures the axial distance along the channel from the continuous line source, while z specifies the position in the cross section. The channel center line coincides with $z = H/2$ so that $0 \leq z \leq H$ where H is the width of the channel.

Before further analysis, the following assumptions are made: first, the variations of the mixture density, velocity, and the dispersion coefficients will not change with the variations of concentration of mass being dispersed. Second, the velocity profile is assumed to be parabolic and can be expressed as:

$$u = 1.5\, U\{1 - \frac{(2z-H)^2}{H^2}\} \tag{4.2}$$

where

U = average fluid velocity.

Using the width, H, as the characteristic length, the nondimensionalized variables of x, z, and C may be written as:

$$\eta = \frac{z}{H} \tag{4.3a}$$

$$X^* = \frac{x/H}{Re\ S_z} \tag{4.3b}$$

$$C^* = \frac{C}{C_o} \tag{4.3c}$$

where

Re = Reynolds number
S_z = Turbulent Schmidt number
ν = kinematic viscosity of water
C_o = initial concentration of mass being transported at the source.

Introducing Eq. (4.3) into Eq. (4.1), the nondimensionalized governing equation then becomes

$$4\eta(1-\eta^2) \frac{\partial C^*}{\partial X^*} = \frac{e_x}{Re^2 S_z^2 e_z} \frac{\partial C^*}{\partial X^{*2}} + \frac{\partial^2 C^*}{\partial \eta^2} \tag{4.4}$$

Let $S_{xz} = \frac{\nu}{(e_x e_z)^{1/2}}$

Eq. (4.4) becomes:

$$4\eta(1-\eta^2) \frac{\partial C^*}{\partial X^*} = \frac{1}{Re^2 S_{xz}^2} \frac{\partial^2 C^*}{\partial X^{*2}} + \frac{\partial^2 C^*}{\partial \eta^2} \tag{4.5}$$

BEGIN LAST LINE OF TEXT
BOTTOM OF IMAGE AREA, OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS

PAGE NUMBER
EPA-287 (Clin.) (4-76)

with the following boundary conditions:

$$\frac{\partial C^*}{\partial \eta} = 0 \qquad \text{at } \eta = 0 \text{ and } \eta = 1 \tag{4.6}$$

Eqs. (4.5) and (4.6) indicate that no mass being transported crosses the boundaries of the channel and that the mixing in the vertical direction is complete at any time. Eq. 4.5 can be further simplified because the product of the Reynolds number and the Schmidt number is generally very large. Therefore, the second term can be neglected.

Eq. (4.5) then becomes

$$4\eta(1-\eta^2)\frac{\partial C^*}{\partial X^*} = \frac{\partial^2 C^*}{\partial \eta^2} \tag{4.7}$$

It clearly indicates that a solution of Eq. (4.7) is a unique one where the concentration distributions are only function of X* and η. There will only be one unique concentration, C* for each X* and η. Since X* is a combination of X, H, Re and S_Z, one can select a value of X* with known concentration distributions along η and determine the location of X which is the sampling location. Thus, one can also determine the number of sampling points required at this location to achieve the accuracy necessary for the sampling.

The effect of the second term in Eq. (4.5) will be studied in this report with some values of the product of Re and S_{XZ}. In general, there will be no significant effect on the results with the product of Re and S_{XZ} over 100.

II. FINITE DIFFERENCE EQUATION

The finite difference[8] is a numerical technique which is generally used to solve partial differential equations. When a function and its derivatives are single-valued, finite and continuous functions of independent variables, the partial differential equation of this function can be approximated by a finite difference equation through Taylor's theorem.

Assume the C* and its derivatives are single-valued, finite and continuous functions of X* and η, then the X*-η plane is subdivided into sets of equal rectangles of sides Δx and $\Delta\eta$ as shown in Figure 3. Let the coordinates X*, η at the mesh point P be:

$$X^* = i\,\Delta x \qquad i = 1, 2, 3, \ldots \tag{4.8a}$$

$$\eta = j\,\Delta\eta \qquad j = 1, 2, 3, \ldots, N \tag{4.8b}$$

Also, denote its corresponding values of concentration at P by C_{ij} and its neighboring points by $C_{i+1,j}$, $C_{i,j+1}$, $C_{i-1,j}$ and $C_{i,j-1}$. With the first and second derivatives of function C* approximated by the forward and

central differences, respectively, Eq. (4.5) is transformed into a finite difference equation as follows:

$$\left\{ \frac{a}{\Delta x} - \frac{b}{(\Delta x)^2} \right\} C_{i+1,j} = \left\{ \frac{a}{\Delta x} - \frac{2b}{(\Delta x)^2} - \frac{2}{(\Delta \eta)^2} \right\} C_{i,j} + \frac{b}{(\Delta x)^2} C_{i-1,j}$$

$$+ \frac{1}{(\Delta \eta)^2} C_{i,j+1} + \frac{1}{(\Delta \eta)^2} C_{i,j-1} \qquad (4.9)$$

where

$$a = 4\eta(1-\eta^2)$$

$$b = \frac{1}{Re^2 S_{xz}^2}$$

Eq. (4.9) is an explicit form of solution which has an error of the order Δx and $(\Delta \eta)^2$; therefore, the accuracy of the solution can be improved by reducing the mesh size of Δx and $\Delta \eta$. Also, Eq. (4.9) can be proved to be stable and convergent as long as the following conditions exist:

$$\frac{1}{a\eta(1-\eta^2)} < 0.5 \qquad (4.10a)$$

$$b << 1 \qquad (4.10b)$$

The boundary conditions of Eq. (4.6) are also transformed into finite difference forms:

$$C_{i,j+1} = C_{i,j-1} \qquad \text{for } j = 1 \qquad (4.11a)$$

$$C_{i,j-1} = C_{i,j=1} \qquad \text{for } j = N \qquad (4.11b)$$

With the known initial and boundary conditions, concentration distributions can be obtained explicitly by solving Eq. (4.9) along the η direction at location X^*. This process is repeated column by column as the distance X^* advances.

III. NUMERICAL SOLUTIONS

As mentioned previously, the accuracy of the finite difference equation depends upon the mesh size chosen for Δx and $\Delta \eta$. However, the percentage error of the numerical solution is not known. To estimate the effect of the mesh size on the accuracy of the numerical solution, appropriate conditions are imposed such that an analytical solution for the partial differential equation is available. Comparison of the two results should reveal the appropriate mesh size required for Δx and $\Delta \eta$ in the numerical solution. To make the analytical solution possible, the second term in Eq. (4.1) is neglected and the initial and boundary conditions are shown in Figure 4.

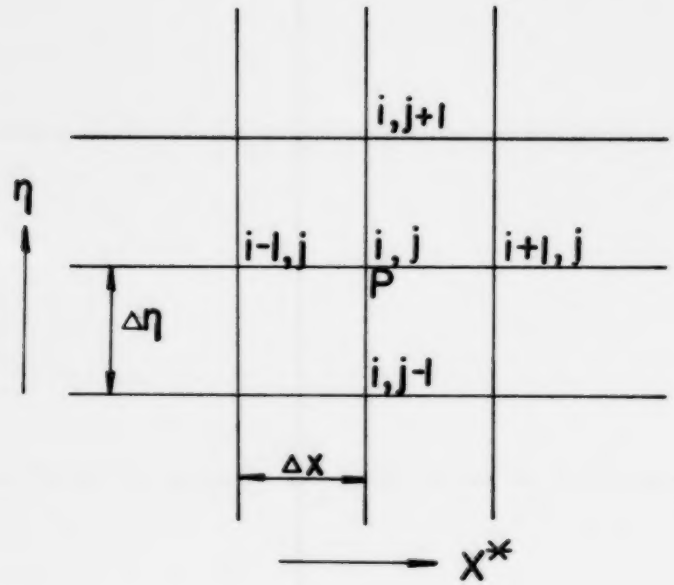

Figure 3. X^*-η plane in a rectangular channel.

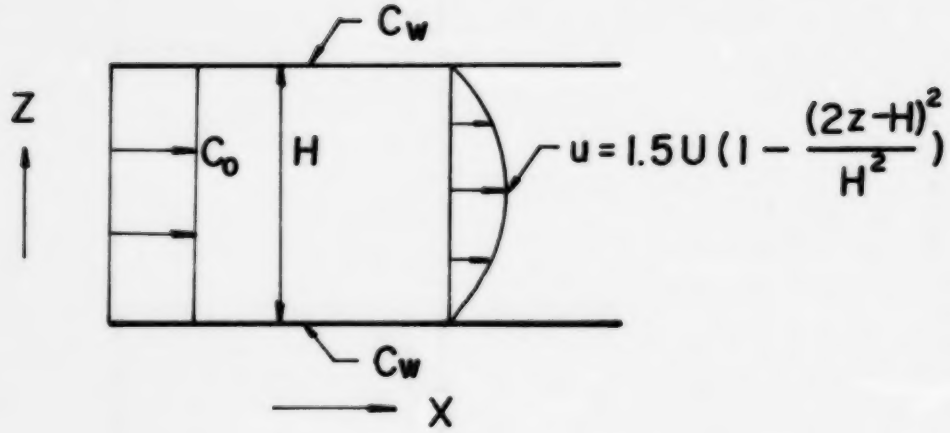

Figure 4. Initial and boundary conditions in a rectangular channel flow.

The velocity profile is assumed to be parabolic and the concentrations at the entrance and the banks are C_o and C_w, respectively. Because of the symmetry of the boundary conditions, the concentration gradient with respect to z in the center line of the channel will be zero. The boundary conditions shown in Figure 5 are written as:

$$\frac{\partial C}{\partial z} = 0 \qquad \text{at } z = 0 \tag{4.12a}$$

$$C = C_w \qquad \text{at } z = \ell \tag{4.12b}$$

$$C = C_o \qquad \text{at } x = 0 \tag{4.12c}$$

Using the length, ℓ, as the characteristic length, the nondimensionalized variables of x, z and C are redefined as:

$$\eta = \frac{z}{\ell} \tag{4.13a}$$

$$X^* = \frac{x/\ell}{Re\ S_z} \tag{4.13b}$$

$$C^* = \frac{C - C_w}{C_o - C_w} \tag{4.13c}$$

Substituting Eqs. (4.13) into Eq. (4.1), the nondimensionalized equation becomes:

$$(1-\eta^2)\frac{\partial C^*}{\partial X^*} = \frac{\partial^2 C^*}{\partial \eta^2} \tag{4.14}$$

with the following conditions, also shown in Figure 6.

$$\frac{\partial C^*}{\partial \eta} = 0 \qquad \text{at } \eta = 0 \tag{4.15a}$$

$$C^* = 0 \qquad \text{at } \eta = 1 \tag{4.15b}$$

$$C^* = 1 \qquad \text{at } X^* = 0 \tag{4.15c}$$

Eq. (4.14) can be solved by separable variables in the following form:

$$C^*(X^*, \eta) = A \exp(-\beta^2 X^*)\psi(\eta) \tag{4.16}$$

where

$$\psi'' + \beta^2(1-\eta^2)\psi = 0 \tag{4.17}$$

The boundary conditions of Eq. (4.17) require that

$$\psi' = 0 \qquad \text{at } \eta = 0 \tag{4.18a}$$

$$\psi = 0 \qquad \text{at } \eta = 1 \tag{4.18b}$$

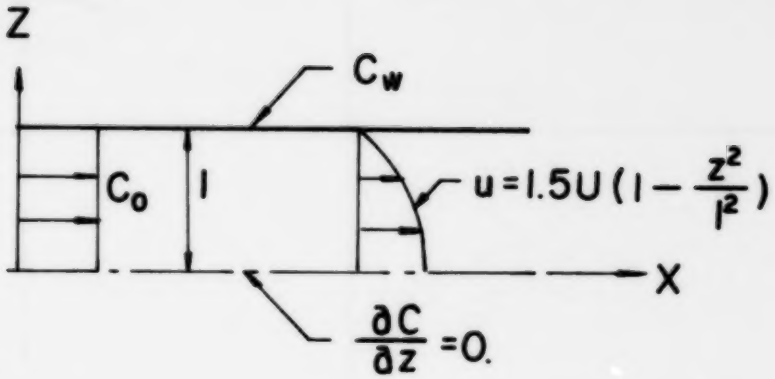

Figure 5. Initial and boundary conditions for the half of a rectangular channel flow.

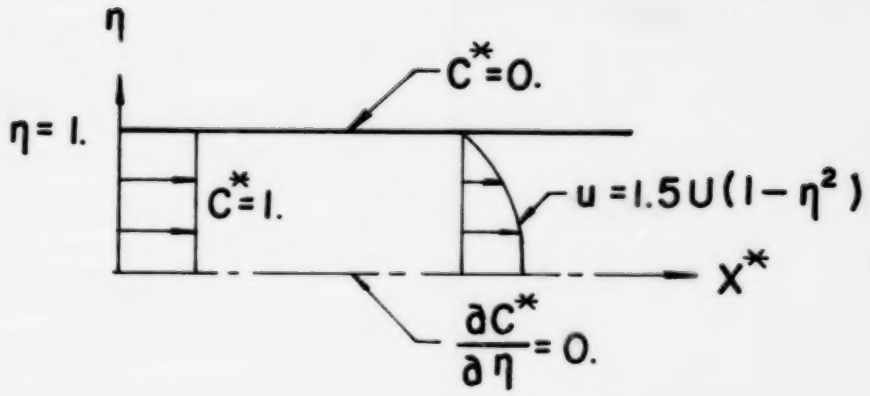

Figure 6. Initial and boundary conditions in nondimensionalized form in a rectangular channel flow.

It is clear that β in Eq. (4.17) plays the role of an eigenvalue. The process of determining each β is one of systematic trial and error. For each trial value of β, Eq. (4.17) is integrated from η=0 to η=1 whereupon Eq. (4.18b) is evaluated and compared with zero. The trial value of β is accepted as an eigenvalue when Eq. (4.18b) is satisfied to within a pre-assigned tolerance. The numerical integration is carried out by the Runge-Kutta method with a step size of $\Delta\eta$=0.01.

Assuming that the eigenvalues have been attained, the solution for the concentration, C*, can be written as:

$$C^*(X^*,\eta) = \sum_{n=1}^{n=\infty} A_n \exp(-\beta_n^2 X^*)\psi_n(\eta) \tag{4.19}$$

with the constant, A_n, undetermined. The A_n can be determined by imposing the entrance condition, Eq. (4.15c) on Eq. (4.19):

$$\sum_{n=1}^{n=\infty} A_n\psi_n(\eta) = 1 \tag{4.20}$$

A standard Sturm-Liouville type orthogonality will suffice to obtain the An as follows:

$$A_n = \frac{\int_0^1 (1-\eta^2)\psi_n(\eta)\, d\eta}{\int_0^1 (1-\eta^2)\psi_n^2(\eta)\, d\eta} \tag{4.21}$$

The integration on the right-hand side of Eq. (4.21) is performed by Simpson's rule. Numerical values of the concentration at any location in the flow can be found directly from Eq. (4.19). The bulk concentration, C_b, may be derived as follows:

$$C_b = \frac{\int_0^\ell uc\, dz}{\int_0^\ell u\, dz} = \int_0^1 \frac{u}{U}\, d\eta \tag{4.22}$$

from which it follows:

$$C_b^* = \frac{C_b - C_o}{C_w - C_o} = 1 + 1.5 \sum_{n=1}^{n=\infty} \frac{A_n\, \psi_n'(1)}{\beta_n^2} \exp(-\beta_n^2 X^*) \tag{4.23}$$

The bulk concentration represents an overall average concentration of the flow. Therefore, it is an indicator to test the accuracy of the numerical method. The values of A_n, β_n and $\psi_n'(1)$ listed in Table 1 are used to calculate C_b^* at any location of X*. Numerical solution under the same conditions are also obtained and are compared in Table 2 with the corresponding values of C_b^* from the analytical solution. The difference between the two results at any location of X* is about 1% or less for the mesh size

$\Delta x = 0.0002$ and $\Delta\eta = 0.05$. Therefore, it is concluded that the above mesh size should be used in the numerical calculation.

TABLE 1. CONSTANTS USED IN EQ. (4.22)

	A_n	β_n	$\psi'_n(1)$
n=1	1.2008324	1.6816025	-1.4291296
n=2	-0.29915899	5.6698599	3.8069992
n=3	0.16082501	9.6682453	-5.9201288
n=4	-0.10743695	13.667671	7.8924608
n=5	0.079646051	17.667404	-9.7707977
n=6	-0.062775254	21.667297	11.579610
n=7	0.05151971	25.667297	-13.333608
n=8	-0.043510258	29.667465	15.042241
n=9	0.037543193	33.667801	-16.712799
n=10	-0.032933667	37.668381	18.349487

Table 2. RESULTS OF BULK CONCENTRATIONS FROM ANALYTICAL AND NUMERICAL SOLUTIONS

Analytical Solution		Numerical Solution $\Delta x=0.001$ $\Delta\eta=0.1$	Numerical Solution $\Delta x=0.0002$ $\Delta\eta=0.05$
X^*	C_b^*	C_b^*	C_b^*
0.01	0.06928	0.06499	0.06904
0.05	0.19890	0.19813	0.19910
0.1	0.31176	0.31974	0.31581
0.2	0.48281	0.50635	0.49423
0.3	0.61126	0.64141	0.62547
0.4	0.70626	0.73951	0.72264
0.5	0.77861	0.81078	0.79459
0.7	0.87424	0.90015	0.88734
1.0	0.94616	0.96172	0.95424
1.5	0.98691	0.99226	0.98981
2.0	0.99682	0.99843	0.99773

Three numerical solutions with different initial conditions are obtained. The mixing distance for each solution is determined from a nondimensionalized distance X^*.

- Concentration distributions for material discharged in the center of the channel with the dispersion neglected in the X-direction

Solutions of concentration distributions at different locations of X^* are obtained through Eq. (4.9) and are shown in Figures 7, 8, and 9 for

material discharged in the center line at the inlet of the channel. It is clear that the concentration distributions are symmetrically distributed at any location of X* and become more and more uniformly distributed as the X* increases. When the ratio of (Cc(at center) - C_∞ (uniform value)) and C_∞ is less than 0.02%, it is found that the X* is equal to 0.14. It is reasonable to assume that this is the mixing distance in nondimensionalized form. In the dimensional form, X_m is

$$X_m = \frac{0.21\ H^2U}{e_z} \tag{4.24}$$

The solution presented here is unique for this condition. The number of sampling points at any location may be determined by Simpson's rule or by any other approximate technique. However, if the mixing distance is chosen, only one sampling point at this location is sufficient.

- Concentration distributions for material discharged at one of the banks with the dispersion neglected in the X-direction

The solution under this condition is shown in Figure 10. The distributions of concentration become uniform at a distance of 0.3. The mixing distance of 0.3 is equivalent to a percentage of 0.5% (also see Eq. 3.30). In dimensional form, it is:

$$X_m = \frac{0.45\ H^2U}{e_z} \tag{4.25}$$

Comparing Eq. (3.30) with Eq. (4.25), it is clear that they are very close. It is interesting to note that the numerical solution gives a unique concentration profile at each location of X* which can be transformed into a form including X, H, Re and S_z.

- Concentration distributions for material discharged at one of the banks with Re S_{xz} = 100

The effect of the dispersion coefficient in the X-direction on the concentration distribution is studied by assuming Re S_{xz} = 100. Concentration distributions at different locations of X* are shown in Figures 11 and 12. Comparing Figure 10 with Figure 12, they are almost identical. The influence of the dispersion in the X-direction can therefore be ignored because the product of Re and S_{xz} is normally much greater than 100.

In summary, the above solutions only demonstrate the versatility of the numerical technique. Almost any concentration distribution and respective mixing distance can be obtained if the initial and boundary conditions are provided.

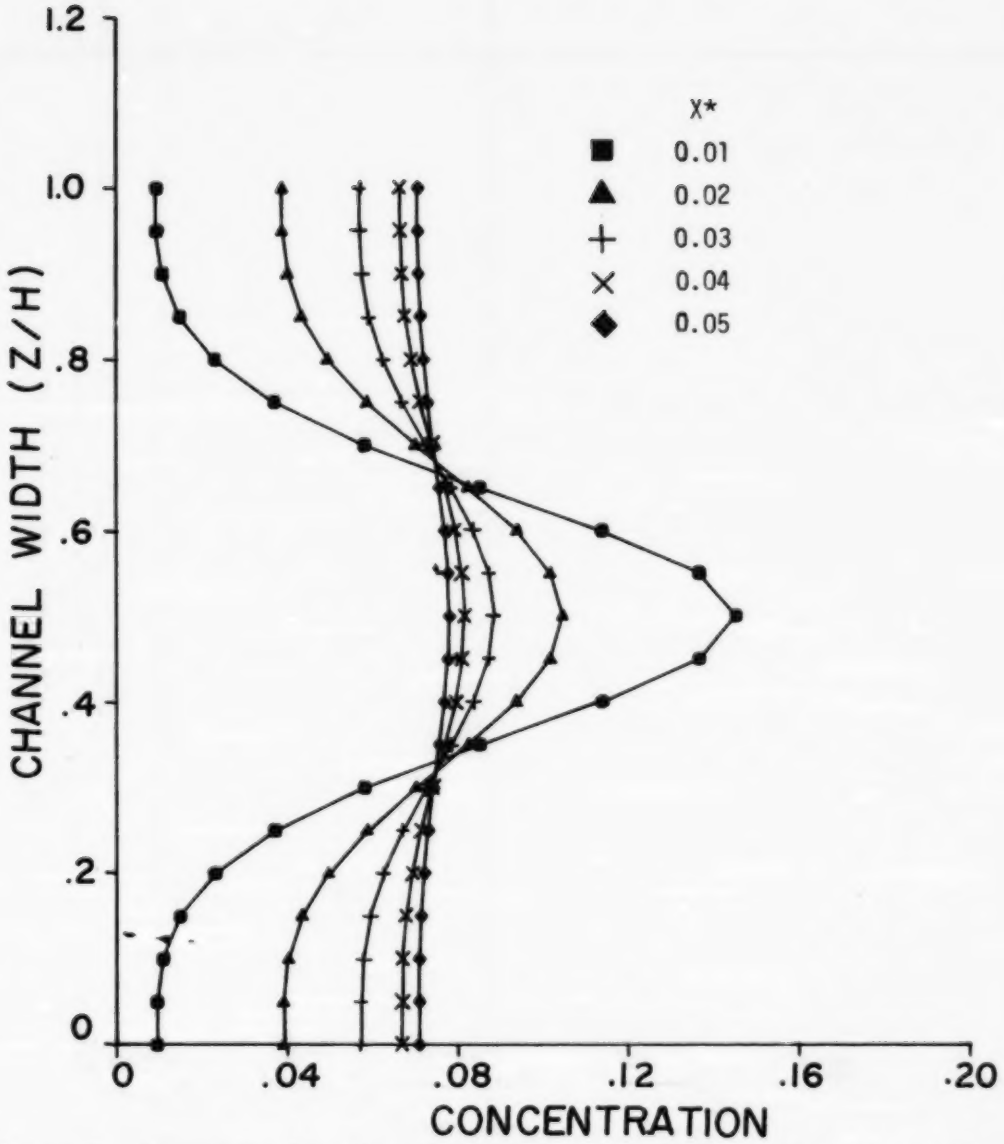

Figure 7. Concentration distributions at X* = 0.01, 0.02, 0.03, 0.04 and 0.05 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.

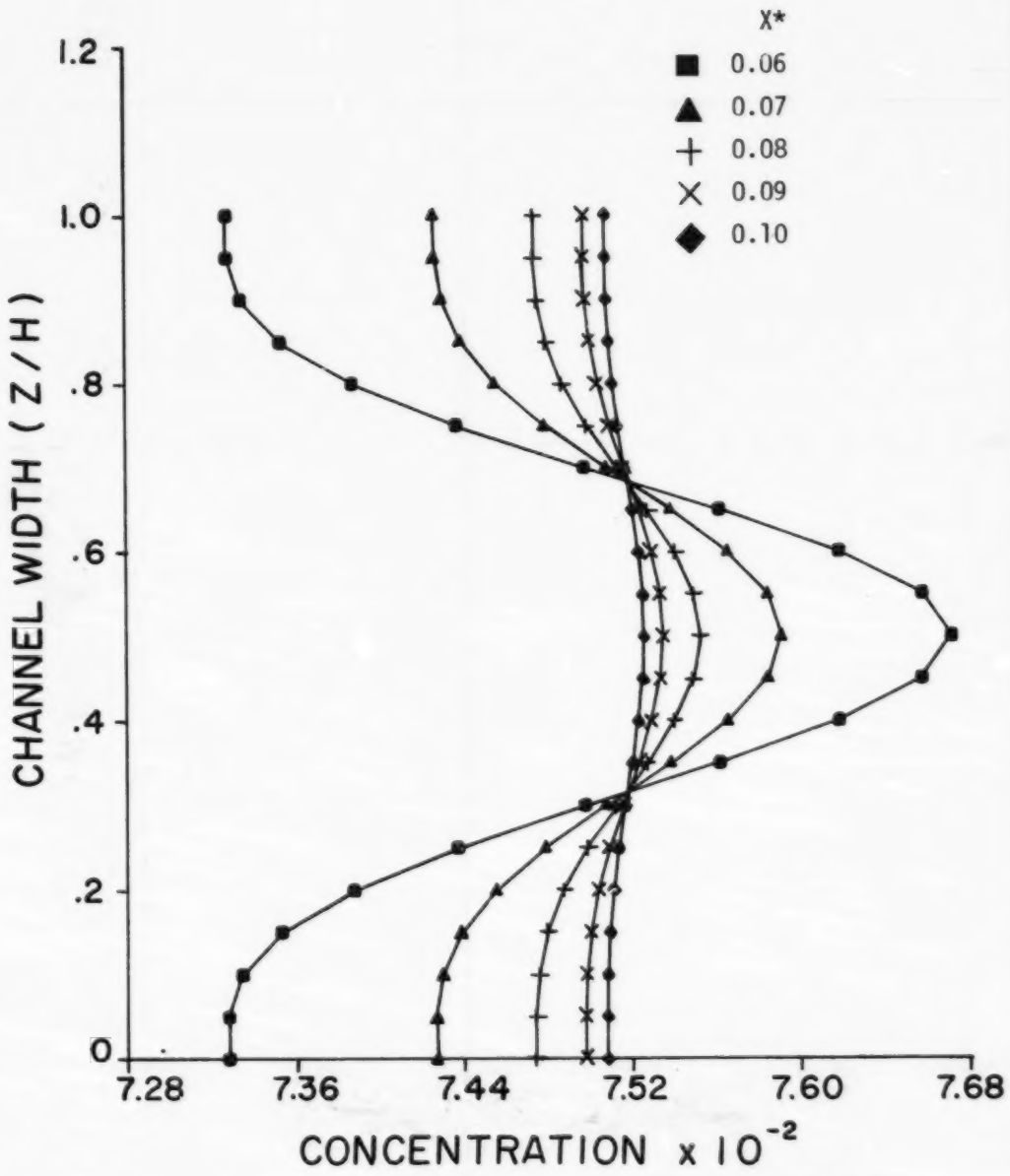

Figure 8. Concentration distributions at X* = 0.06, 0.07, 0.08, 0.09 and 0.1 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.

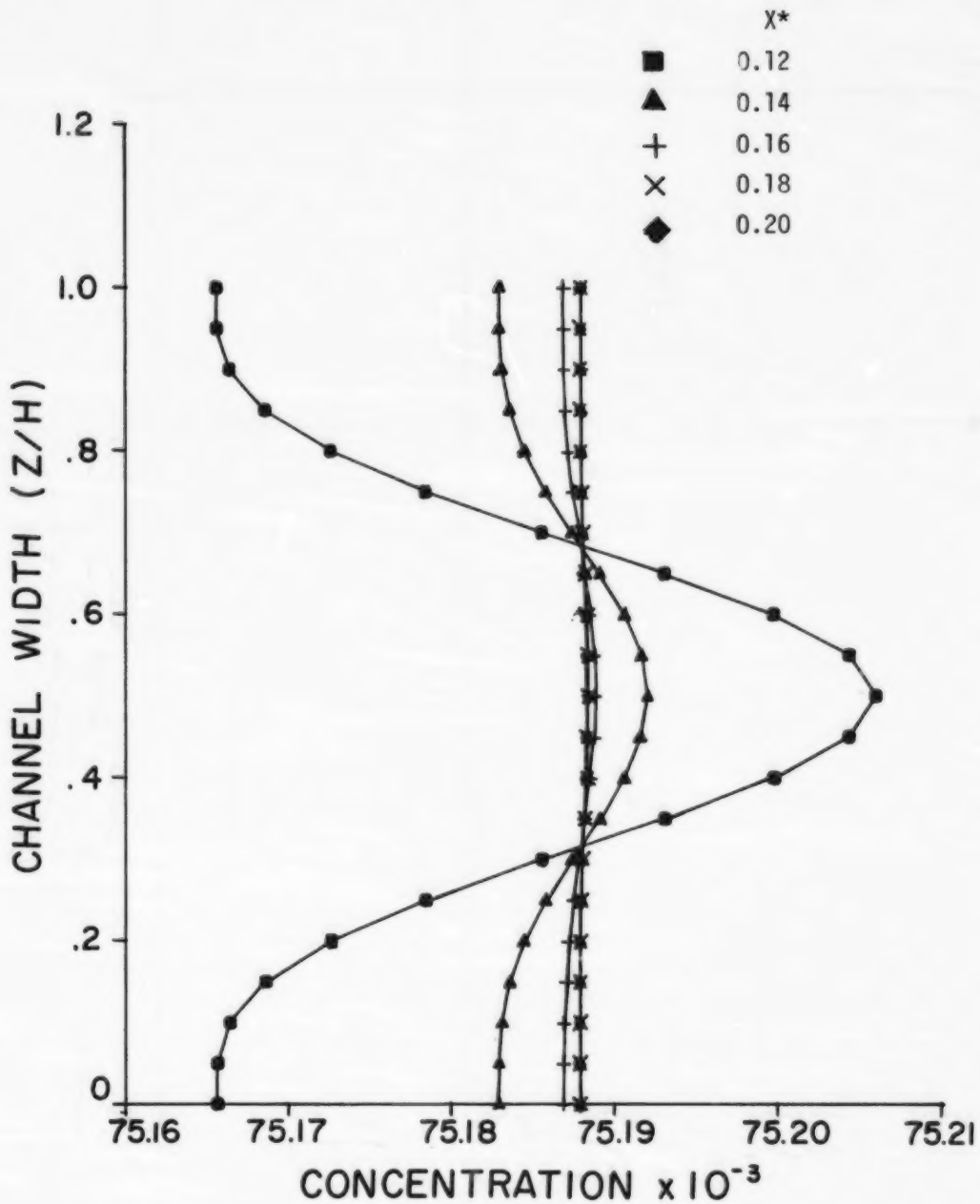

Figure 9. Concentration distributions at X* = 0.12, 0.14, 0.16, 0.18 and 0.20 from a line source located at X* = 0 and η = 0.5 with Re S_{xz} >> 100.

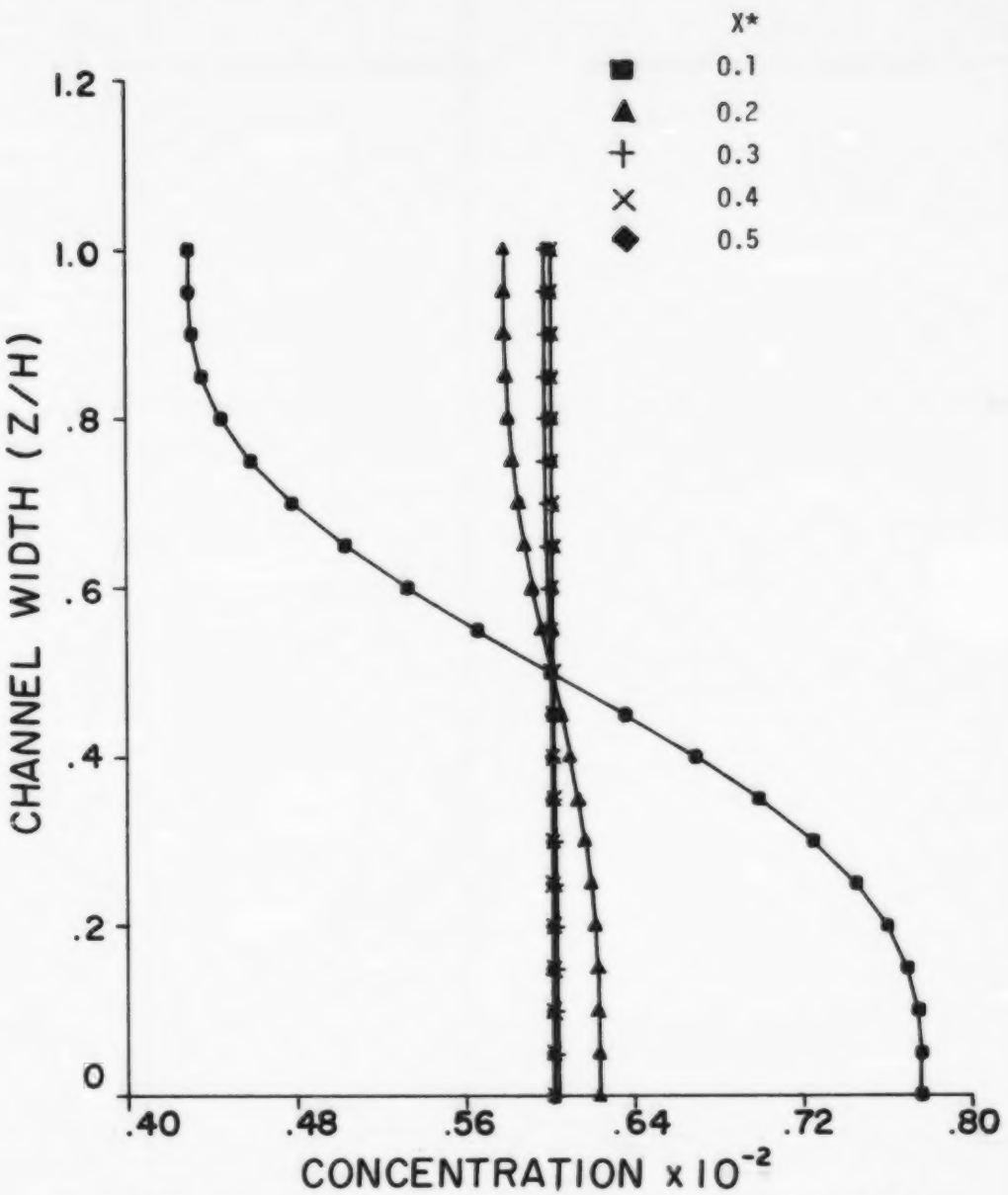

Figure 10. Concentration distribution at X* = 0.1, 0.2, 0.3 0.4 and 0.5 from a line source located at X* = 0 and η = 0 with Re S_{xz} >> 100.

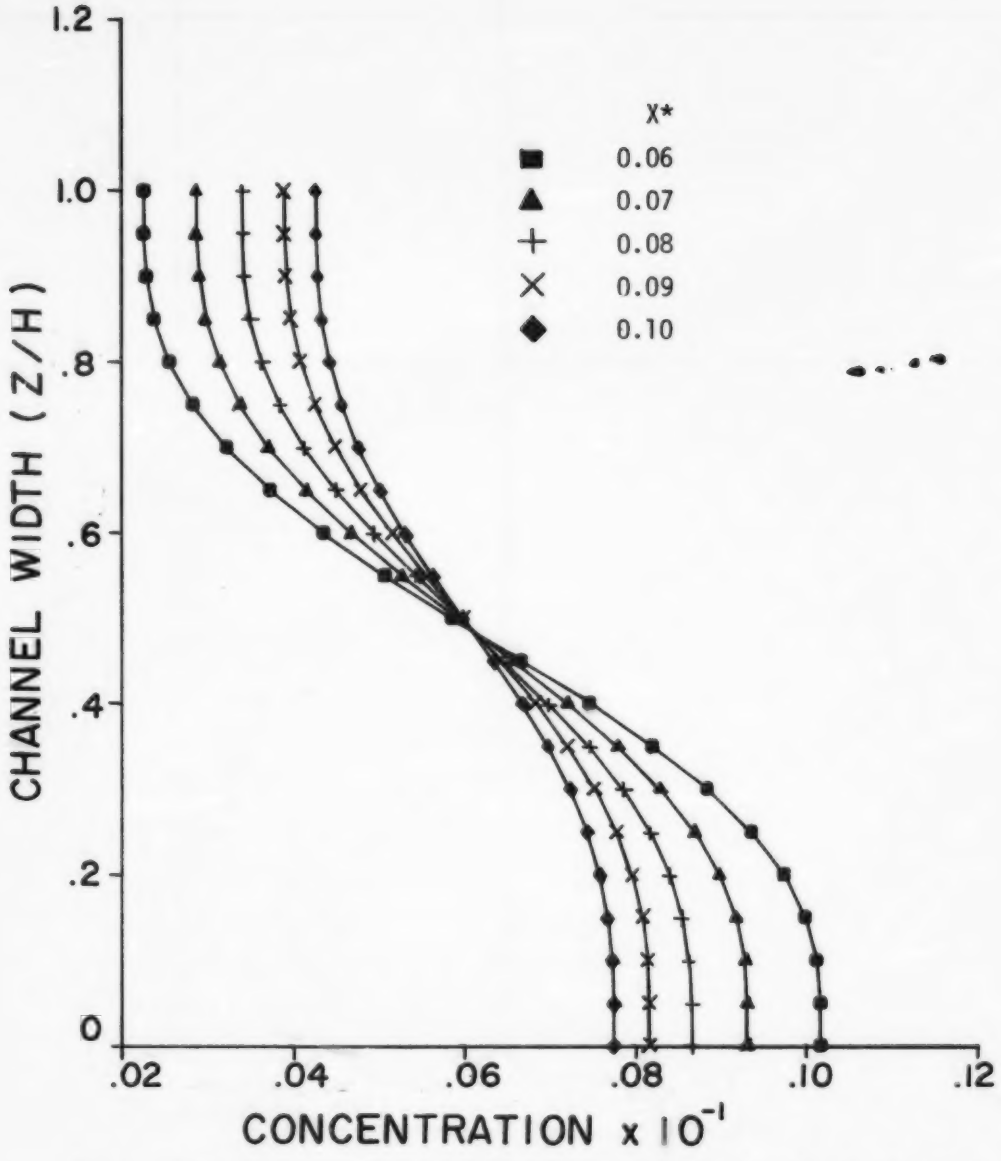

Figure 11. Concentration distributions at X* = 0.06, 0.07, 0.08, 0.09 and 0.1 from a line source located at X* = 0 and η = 0 with Re S_{xz} = 100.

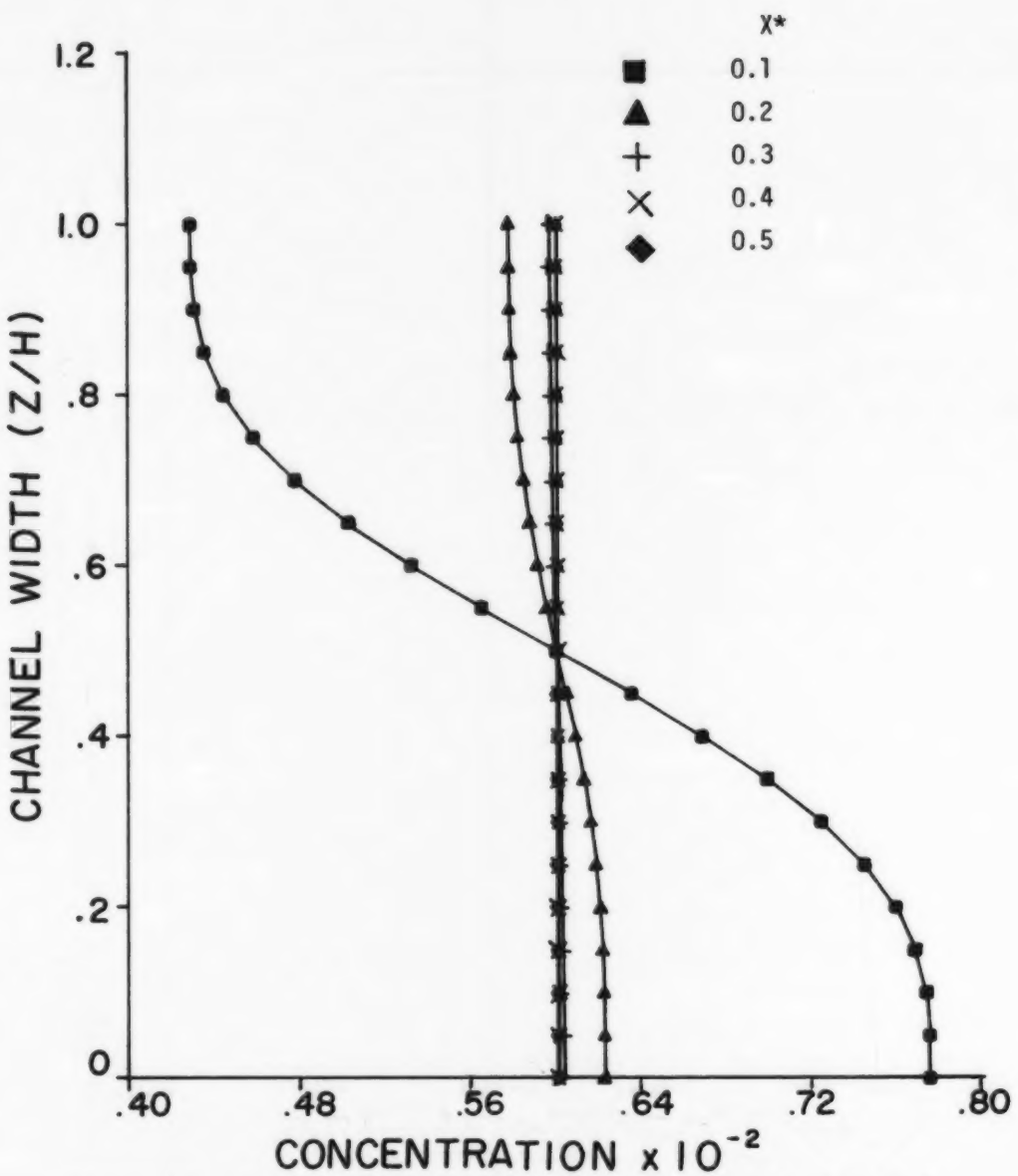

Figure 12. Concentration distributions at X* = 0.1, 0.2, 0.3, 0.4 and 0.5 from a line source located at X* = 0 and η = 0 with Re S_{xz} = 100.

BEGIN FIRST LINE OF TEXT HERE
TYPING GUIDE SHEET
CENTER OF PAGE
TOP OF IMAGE AREA
DROPPED HEAD BEGIN SECTIONS HERE

SECTION 5

SAMPLING FREQUENCY

As discussed in Section 2, mass dispersion in a steady-state stream from a continuous source eventually reaches a steady-state condition. Because of the nonuniformity of concentration distribution in a cross section, the measurement of concentration must be conducted across the water channel at a location beyond the mixing distance. For nonsteady-state conditions of flow and mass dispersion, the selection of a sampling location and sampling frequency is more difficult. In general, a composite sample in the complete mixing zone is desired to minimize the sampling frequency and still preserve the accuracy required for the sample. Samples may be composited by one of the following methods:

(a) A fixed volume of sample collected at constant intervals is composited in a single container.

(b) The volume of sample collected is proportional to the flow rate between constant intervals.

(c) A fixed volume of sample collected at constant intervals is held in each respective container. A composite sample is then collected by extracting a flow-proportional sample from each sample container.

(d) A fixed volume of sample is collected after a certain quantity of flow has passed the sampling station betwen sampling intervals.

The accuracy of concentration from composite samples as compared to actual average concentration depends upon the sampling frequency and the composite method. In mathematical forms, determination of composite concentration can be expressed according to methods (a), (b), (c) and (d) as follows:

$$\bar{C} = \frac{\Sigma C_i}{m} \quad \text{(constant sampling time interval and fixed sample volume)} \quad (5.1)$$

$$\bar{C} = \frac{\Sigma Q_i C_i}{\Sigma Q_i} \quad \text{(constant sampling time interval and varied sample volume)} \quad (5.2)$$

$$\bar{C} = \frac{\Sigma \Delta Q_i C_i}{\Sigma \Delta Q_i} \quad \text{(constant sampling time interval and varied sample volume)} \quad (5.3)$$

$$\bar{C} = \frac{\Sigma C_i}{m} \quad \text{(Fixed quantity of flow and sample volume)} \quad (5.4)$$

BEGIN LAST LINE OF TEXT
3/8"
BOTTOM OF IMAGE AREA OUTSIDE DIMENSION FOR TABLES AND ILLUSTRATIONS

PAGE NUMBER
EPA-267 (Cin.)
(4-76)

where

C_i = concentration at the i^{th} sample
Q_i = flow rate at the i^{th} sample
ΔQ_i = flow volume between $(i-1)^{th}$ and i^{th} samples
m = sampling frequency

The actual average concentration is similar to the bulk concentration. Eq. (4.22) and can be expressed as

$$C_a = \frac{\int_0^T QC\, dt}{\int_0^T Q\, dt} \tag{5.5}$$

where

T = period of a flow cycle
Q = flow rate
t = time

The ratio between the composite and actual average concentrations determines the sample accuracy for each composite method. While continuous flow-proportional sampling may not be feasible, Shelley and Kirkpatrick[10] studied the above four composite techniques by using a sampling frequency m=10 and normalizing time and concentration so that time is ≤ 1 and concentration is ≤ 1. The ratio of the composite sample concentration to actual concentration is shown in Table 3 for selected artificial flow and concentration patterns. The result in the table indicates that method(a) is the worst and method(b) seems to be the best among the four methods. In general, the last three techniques provide an accuracy exceeding 90% for all flow and concentration patterns at m=10.

More examples on the same subject are explored in this study for determining the possible minimum sampling frequency required for maintaining necessary accuracy of the samples being collected.

EXAMPLE 1. Nondimensionalized flow rate and concentration patterns are assumed to be two sine curves with different amplitudes, frequencies and phase angles and are shown as:

$$Q = a \sin(wt)+1 \tag{5.6}$$

$$C = b \sin(rwt+\phi)+1 \tag{5.7}$$

where

a, b = amplitudes of flow rate and concentration, respectively.
r = ratio of frequency between flow rate and concentration.
ϕ = phase angle
w = radian per unit time

By varying a, b, r and ϕ, many flow rate and concentration patterns are created. Results from the four composite methods for different combinations of a, b, r and ϕ are shown in Tables 4, 5, 6 and 7. Results indicate that

as long as the values of a and b are less than or equal to 0.5, the accuracy of the concentration is more than 94% for the last three methods at a sampling frequency of m=8. For values of a and b equal to 1.0, the concentration still reaches a 90% or above accuracy. The worst among the four methods is the first one from which a 60% accuracy is observed for r = 0.2 and 0.5.

EXAMPLE 2

Field measured data on trihalomethanes concentration from Smith et al.[11] are shown in Figure 13 and are recorded in Table 8. An artificial flow pattern as Eq. (5.6) is assumed and the whole sampling period is considered as a cycle for the flow.

Results from the four methods are listed in Table 9 for m=8, 10, 12 and 24. They show an accuracy of 97% or above even at m=8 for the four methods.

In summary, it may be reasonably concluded that in a completely mixed zone, a minimum sampling frequency of m=8 seems to be sufficient to obtain representative samples in most cases for the last three composite techniques.

Table 3. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION*

Flow		Concentration				
		$1-t$	$1-\frac{t}{2}$	$\cos \frac{\pi t}{2}$	e^{-t}	$\sin \pi t$
Constant	(a)	0.90	0.97	0.92	0.95	0.99
	(b)	0.90	0.97	0.92	0.95	0.99
	(c)	0.90	0.97	0.92	0.95	0.99
	(d)	0.90	0.97	0.92	0.95	0.99
t	(a)	1.35	1.09	1.26	0.14	0.99
	(b)	0.90	0.97	0.90	0.97	0.90
	(c)	0.86	0.96	0.87	0.95	0.89
	(d)	0.87	0.96	0.89	0.95	0.97
$1-t$	(a)	0.68	0.87	0.72	0.82	0.99
	(b)	0.95	0.98	0.98	0.96	1.12
	(c)	0.92	0.97	0.95	0.95	1.09
	(d)	0.92	0.97	0.93	0.95	0.97
$\sin \pi t$	(a)	0.90	0.97	0.88	0.97	0.80
	(b)	1.01	1.00	1.00	1.00	1.01
	(c)	0.90	0.97	0.98	0.95	0.98
	(d)	0.90	0.97	0.92	0.95	0.97

(a) Equation 5.1
(b) Equation 5.2
(c) Equation 5.3
(d) Equation 5.4

*See Reference 10.

Table 4. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION IN EXAMPLE 1 (METHOD 1)

		m = 8			m = 12			m = 24		
		a=1.0	1.0	0.5	1.0	1.0	0.5	1.0	1.0	0.5
		b=1.0	0.5	0.5	1.0	0.5	0.5	1.0	0.5	0.5
r=0.2	$\phi=0$	1.155	1.09	1.055	1.14	1.082	1.048	1.127	1.074	1.040
	$\phi=\pi/2$	.915	.945	.964	.923	.950	.969	.931	.955	.975
	$\phi=\pi$	.645	.866	.910	.676	.878	.923	.708	.889	.936
r=0.5	$\phi=0$	.995	.997	.997	.998	.999	.999	.999	.999	.999
	$\phi=\pi/2$	.614	.773	.848	.644	.791	.866	.673	.808	.885
	$\phi=\pi$	1.023	1.006	1.006	1.010	1.003	1.003	1.001	1.001	1.001
r=1.0	$\phi=0$	.677	.800	.889	.667	.800	.889	.667	.800	.889
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	2	1.333	1.143	2	1.333	1.143	2	1.333	1.143
r=2.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1
r=5.0	$\phi=0$	.999	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1

NOTE:
m = sampling frequency
a = amplitude of flow rate
b = amplitude of concentration
ϕ = phase angle
r = frequency ratio

Table 5. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION IN EXAMPLE 1 (METHOD 2)

		m = 8			m = 12			m = 24		
		a=1.0 b=1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5
r=0.2	$\phi=0$	1.048	1.028	1.025	1.031	1.018	1.017	1.015	1.009	1.008
	$\phi=\pi/2$	.973	.982	.983	.983	.989	.989	.992	.995	.995
	$\phi=\pi$	.891	.959	.959	.930	.974	.973	.966	.987	.987
r=0.5	$\phi=0$	.995	.997	.997	.998	.999	.999	.999	1.000	1.000
	$\phi=\pi/2$	.901	.942	.940	.936	.963	.961	.970	.982	.981
	$\phi=\pi$	1.023	1.006	1.006	1.010	1.003	1.003	1.003	1.001	1.001
r=1.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1
r=2.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1
r=5.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1

Table 6. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION IN EXAMPLE 1 (METHOD 3)

		m = 8			m = 12			m = 24		
		a=1.0 b=1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5
r=0.2	$\phi=0$	1.049	1.028	1.026	1.033	1.019	1.017	1.016	1.010	1.009
	$\phi=\pi/2$	.982	.989	.986	.989	.993	.991	.995	.996	.996
	$\phi=\pi$	.888	.958	.959	.925	.972	.974	.963	.986	.986
r=0.5	$\phi=0$	1.045	1.028	1.013	1.024	1.015	1.007	1.016	1.010	1.005
	$\phi=\pi/2$	.905	.944	.941	.954	.973	.971	.970	.982	.981
	$\phi=\pi$	.796	.946	.976	.891	.971	.986	.926	.980	.990
r=1.0	$\phi=0$	.967	.980	.989	.985	.991	.995	.996	.998	.999
	$\phi=\pi/2$	.814	.907	.953	.872	.936	.968	.935	.968	.984
	$\phi=\pi$	1.100	1.033	1.014	1.045	1.015	1.006	1.011	1.004	1.002
r=2.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1
r=5.0	$\phi=0$	1	1	1	1	1	1	1	1	1
	$\phi=\pi/2$	1	1	1	1	1	1	1	1	1
	$\phi=\pi$	1	1	1	1	1	1	1	1	1

Table 7. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION IN EXAMPLE 1 (METHOD 4)

		m = 8			m = 12			m = 24		
		a=1.0 b=1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5	1.0 1.0	1.0 0.5	0.5 0.5
r=0.2	$\phi=0$	1.038	1.022	1.024	1.032	1.018	1.016	1.015	1.0099	1.008
	$\phi=\pi/2$	.979	.987	.984	.981	.988	.989	.991	.994	.995
	$\phi=\pi$	.913	.967	.962	.928	.973	.974	.965	.987	.987
r=0.5	$\phi=0$	1.004	1.002	.997	.991	.994	.997	.997	.1998	.999
	$\phi=\pi/2$	.924	.955	.943	.931	.959	.962	.967	.981	.981
	$\phi=\pi$	.984	.996	1.006	1.041	1.011	1.001	1.015	1.004	1.001
r=1.0	$\phi=0$	1.006	1.003	1.001	.993	.006	1.000	1.000	1.000	1
	$\phi=\pi/2$	.957	.978	1.001	1.037	1.018	1	1.013	1.006	1
	$\phi=\pi$	.983	.994	.999	1.020	1.007	1.000	1.001	1.000	1
r=2.0	$\phi=0$	1.075	1.038	.995	.930	.965	1.000	.974	.987	1
	$\phi=\pi/2$	1.026	1.013	1.004	.969	.985	.999	.999	.999	1
	$\phi=\pi$	.925	.962	1.005	1.070	1.035	1.000	1.026	1.013	1
r=5.0	$\phi=0$	.946	.973	1.088	1.082	1.041	.974	1.009	1.004	1.000
	$\phi=\pi/2$	1.117	1.09	1.098	.987	.994	1.008	1.013	1.006	1
	$\phi=\pi$	1.054	1.027	.912	.918	.959	1.026	.991	.995	1.000

Table 8. RECORDED CONCENTRATIONS OF TOTAL TRIHALOMETHANES IN DRINKING WATER

	Time	Concentration (ppb)		Time	Concentration (ppb)
Oct. 22	10	105	Oct. 26	2	80
	14	117		6	91
	18	103		10	104
	22	100		14	90
Oct. 23	2	90		18	70
	6	126		22	95
	10	130	Oct. 27	2	92
	14	128		6	106
	18	116		10	130
	22	108		14	90
Oct. 24	2	107		18	81
	6	105		22	97
	10	108	Oct. 28	2	96
	14	95		6	101
	18	98		10	120
	22	92		14	100
Oct. 25	2	76		18	84
	6	83		22	71
	10	88	Oct. 29	2	101
	14	89		6	83
	18	95			
	22	85			

Table 9. RATIO OF COMPOSITE SAMPLE CONCENTRATION TO ACTUAL CONCENTRATION IN EXAMPLE 2

Method	m=8		m=10		m=12		m=24	
	a=0.5	a=1.0	a=0.5	a=1.0	a=0.5	a=1.0	a=0.5	a=1.0
1	0.9869	0.9736	0.9600	0.9471	0.9767	0.9636	0.9817	0.9685
2	0.9987	0.9968	0.9635	0.9540	0.9889	0.9877	0.9971	0.9988
3	0.9858	0.9715	0.9616	0.9503	0.9876	0.9844	0.9947	0.9943
4	0.9767	0.9820	1.0014	0.9727	0.9841	0.9989	0.9833	0.9912

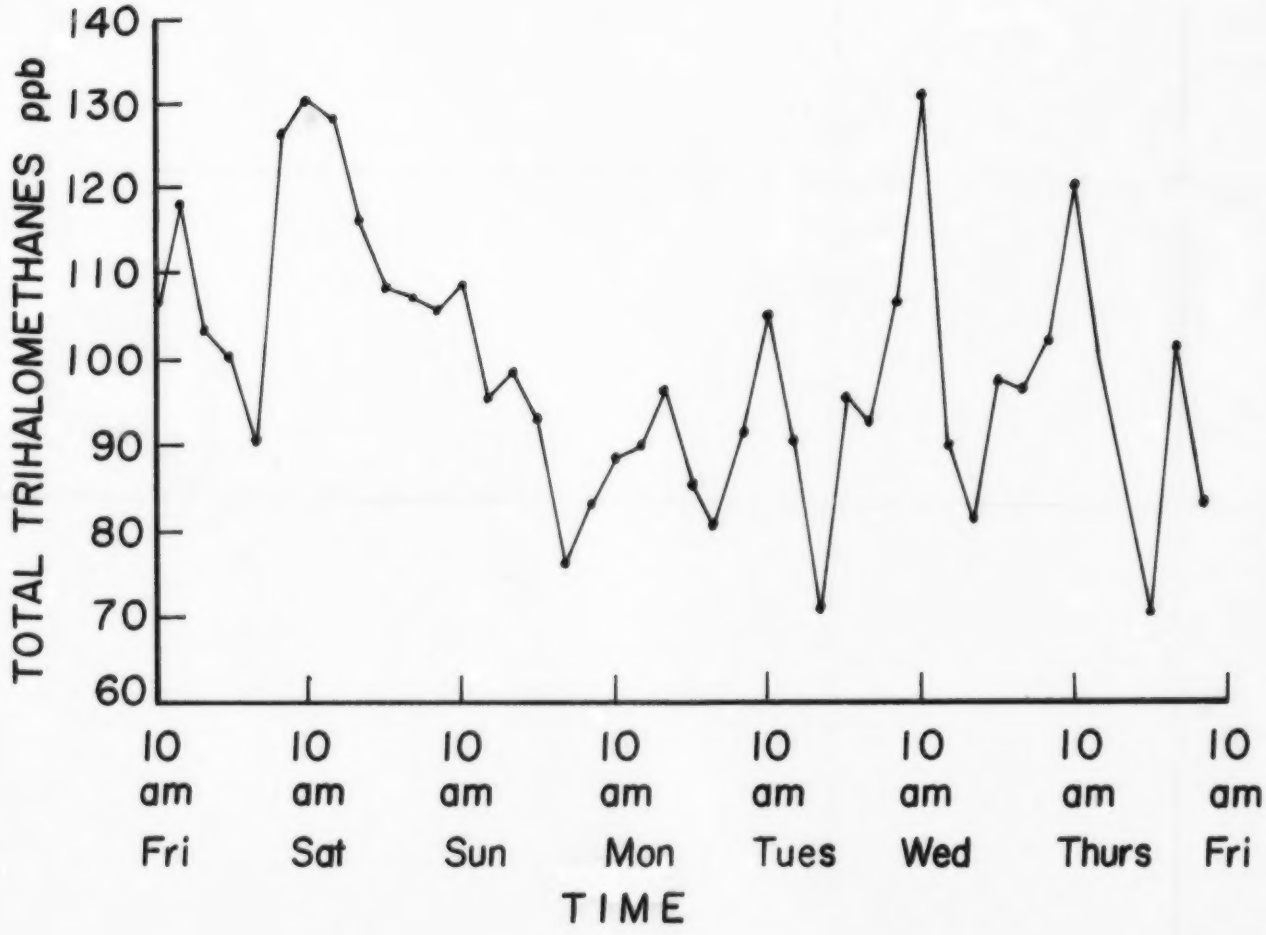

Figure 13. Concentrations of total trihalomethanes in drinking water (see reference 11).

REFERENCES

1. Sayre, W.W. Natural Mixing Processes in Rivers. In: Environmental Impact on Rivers, H.W. Shen, ed. Department of Civil Engineering, Colorado State University, Fort Collins, Colorado, 1973.

2. Carter, H.H. and A. Okubo. A Study of the Physical Processes of Movement and Dispersion in the Cape Kennedy Area. Final Report, U.S. Atomic Energy Commission Contract No. AT(30-1)-2973, 1965.

3. Holley, E.R. Transverse Mixing in Rivers. Report No. S132, Delft Hydraulic Laboratory, Delft, The Netherlands, 1971.

4. Sayre, W.W. and F.M. Chang. A Laboratory Investigation of Open-Channel Dispersion Processes for Dissolved, Suspended and Floating Dispersants. U.S. Geological Survey Prof. Paper 433-E, 1968.

5. Monin, A.S. and A.M. Yaglom. Statistical Fluid Mechanics, Massachusetts Institute of Technology Press, Cambridge, 1971.

6. Carslaw, H.S. and J.C. Jaeger. Conduction of Heat in Solids. The Clarendon Press, Oxford, Great Britain, 1959.

7. Yotsukura, N. and E.D. Cobb. Transverse Diffusion of Solutes in Natural Streams. U.S. Geological Survey Prof. Paper 582-C, 1972.

8. Smith, G.C. Numerical Solution of Partial Differential Equations. Oxford University Press, New York, 1965.

9. Conte, S.D. Elementary Numerical Analysis. McGraw-Hill Book Company, New York, 1965.

10. Shelley, P.E. and G.A. Kirkpatrick. An Assessment of Automatic Sewer Flow Samplers - 1975. EPA-600/2-75-065, U.S. Environmental Protection Agency, Cincinnati, Ohio 45268.

11. Smith, V.L., I. Cech, J.H. Brown and G.F. Bogdan. Temporal Variations in Trihalomethanes Content of Drinking Water. Enviro. Sci. and Tech., Vol. 14, No. 2, 1980.